Diffusive proton acceleration and injection efficiency at shocks of arbitrary speed

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1. Introduction

Apart from magnetic reconnection in transient electric fields cosmic-ray particles in magnetized cosmic plasmas are accelerated by first- and second-order Fermi processes interacting with low-frequency electromagnetic fluctuations such as Alfven and Whistler plasma waves. First-order Fermi acceleration at shock waves is a prime candidate for particle acceleration in astrophysics process because of the large kinetic energy reservoir of supersonic outflows.

Shocks occur ubiquitously in space due to violent events such as supernova explosions and supersonic winds and jets. Magnetized shock waves form due to the interaction of the supersonic and super-Alfvenic outflows with the ambient ionized interplanetary, interstellar or intergalactic medium.

Pulsar wind nebulae, active galactic nuclei and gamma-ray bursts exhibit highly collimated winds or jets with initial relativistic bulk Lorentz factors $\Gamma_0 = (1 - (V_0/c)^2)^{-1/2}$ up to $100 - 10^3$. Therefore there is high interest to understand the acceleration of cosmic rays at magnetized shocks of arbitrary speed.

Axford, Leer and Skadron (1977), Krymskii (1977), Bell (1978) and Blandford and Ostriker (1978) developed the diffusive first-order Fermi acceleration theory at nonrelativistic shock fronts (for reviews see Drury 1983; Blandford and Eichler 1987). A fully analytical theory of diffusive first-order Fermi acceleration theory at relativistic shock fronts is still not available, although Kirk and Schneider (1987) pioneered a semi-analytical eigenfunction method. Monte-Carlo simulations and particle-in-cell simulations are much in use (for recent reviews see Sironi, Keshet and Lemoine 2015 and Marcowith et al. 2016)

Vietri (2003) has noted that "the theory of particle acceleration at shocks is currently unsatisfactory" as one cannot follow this process from first principles. Moreover he emphasized that even in the test-particle limit "there is no general equation for the particle distribution function for arbitrary (i.e., even relativistic) shock speed, and no general treatment is available when the distribution function is anisotropic". His criticism still is true today.

To remedy these deficiencies RS (2015) and Antecki, RS and Krakau (2016) started the derivation of the analytical theory of diffusive particle acceleration at parallel shocks of arbitrary speed, which is further developed here.

2. First principles

To keep the mathematical complexities at a minimum, we consider the case of an infinitely extended planar shock with the step-like shock profile, sketched in Fig. 1, with positive $0 < \beta_d c < \beta_u c$.

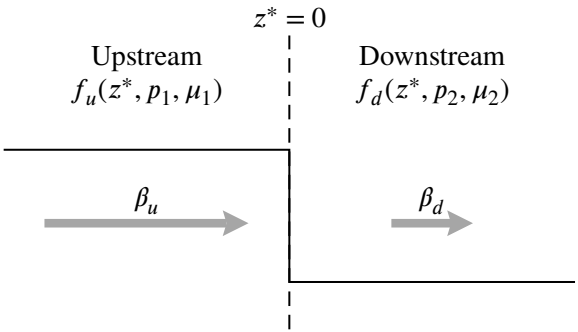

Figure 1: Sketch of the considered parallel relativistic shock wave in the shock rest system.

In order to automatically remove the strong particle momentum anisotropy due to the relativistically moving medium, we take the cosmic-ray phase space coordinates in the mixed comoving coordinate system: time t^* and space coordinates z^* in the laboratory (=shock rest) system and particle's momentum coordinates p and $\mu = p_\parallel/p$ in the rest frame of the streaming plasma.

This choice then also allows us to apply the diffusion approximation to cosmic-ray transport in the respective up- and downstream media. Let (p, μ) and f represent either (p_1, μ_1) and f_u or (p_2, μ_2) and f_d, respectively. It is also useful to keep the parallel momentum coordinate $p_\parallel = p\mu$. It is essential that, because of the chosen mixed comoving coordinate system, the up- (p_1, μ_1) and downstream (p_2, μ_2) cosmic-ray particle momenta in general are different.

In both reference systems the Larmor-phase averaged steady-state Fokker-Planck transport equation (without momentum losses) for the anisotropic but gyrotropic cosmic-ray phase space density $f(z^*, p, \mu)$ in a medium, propagating with the stationary bulk speed $\vec{U} = U(z)\vec{e_{z^*}} = \beta_s c \vec{e_{z^*}}$ with $\Gamma = [1 - \beta_s^2]^{-1/2}$ aligned along the magnetic field direction, reads (Lindquist 1966; Webb 1985; Kirk, Schneider and Schlickeiser 1988)

$$\boxed{\Gamma\left[U + v\mu\right]\left[\frac{\partial f_0}{\partial z^*} - \frac{dU}{dz^*}\Gamma^2\frac{p}{v}\frac{\partial f_0}{\partial p_\parallel}\right] = Q(z^*, p, \mu) + \frac{\partial}{\partial \mu}\left[D_{\mu\mu}(\mu)\frac{\partial f_0}{\partial \mu}\right]}, \quad (1)$$

where

$$Q(z^*, p_1, \mu_1) = Q_0(p_1, \mu_1)\delta(z^*) \quad (2)$$

is the assumed particle injection rate solely at the position of the shock. With

$$\frac{d(\Gamma U)}{dz^*} = \Gamma^3\frac{dU}{dz^*} \quad (3)$$

the transport equation (1) can be written as

$$\Gamma\left[U+v\mu\right]\frac{\partial f_0}{\partial z^*}-\frac{d(U\Gamma)}{dz^*}(U+v\mu)\frac{p}{v}\frac{\partial f_0}{\partial p_\parallel}=Q_0(p_1,\mu_1)\delta(z^*)+\frac{\partial}{\partial\mu}\left[D_{\mu\mu}(\mu)\frac{\partial f_0}{\partial\mu}\right] \tag{4}$$

With dimensionless momentum coordinates

$$y=\frac{p}{mc},\quad y_\parallel=\frac{p_\parallel}{mc},\quad y_\perp=\frac{p_\perp}{mc},\quad \mu=\frac{y_\parallel}{\sqrt{y_\parallel^2+y_\perp^2}},\quad y=\sqrt{y_\parallel^2+y_\perp^2}, \tag{5}$$

Eq. (4) after a few manipulations can be written in the form

$$\begin{aligned}\frac{\partial S(z^*,y,y_\parallel)}{\partial z^*}&+\Gamma\beta_s c\frac{\partial}{\partial z^*}\frac{\partial}{\partial y_\parallel}[(\beta_s\sqrt{1+y^2}+y_\parallel)f_0(z^*,y,y_\parallel)]\\&=Q_0(y_1,\mu_1)\delta(z^*)+\frac{\partial}{\partial\mu}\left[D_{\mu\mu}(\mu)\frac{\partial f_0(z^*,y,y_\parallel)}{\partial\mu}\right]\end{aligned} \tag{6}$$

with the cosmic-ray current

$$\boxed{S(z^*,y,y_\parallel)=\frac{cy_\parallel f_0}{\Gamma\sqrt{1+y^2}}-c\beta_s\Gamma(\beta_s\sqrt{1+y^2}+y_\parallel)\frac{\partial f_0}{\partial y_\parallel}} \tag{7}$$

Eqs. (6) -(7) are the first important new results of our investigation as they determine the momentum spectrum of accelerated cosmic-ray particles at shocks of arbitrary velocity. Eq. (6) represents the generalization of the transport equation of Gleeson and Axford (1967) accounting for shocks of arbitrary speed and anisotropic but gyrotropic cosmic-ray particle distribution functions. For non-relativistic shocks Gleeson and Axford (1967) showed that, apart from sources, the cosmic-ray "jump" conditions at a shock front are simply that the isotropic phase space density and the isotropic current

$$F(y, z^*) = \frac{1}{2}\int_{-1}^{1} d\mu f_0(y, \mu, z^*), \quad \bar{S}(y, z^*) = \frac{1}{2}\int_{-1}^{1} d\mu S(y, \mu, z^*) \tag{8}$$

are continuous.

2.1. Transport equations in the up- and down-stream medium

For the step-like shock velocity profile the rate of adiabatic acceleration (3)

$$\frac{d(U\Gamma)}{dz^*} = -(\Gamma_u\beta_u - \Gamma_d\beta_d)c\delta(z^*), \tag{9}$$

so that the Fokker-Planck transport equation (4) in the up- ($z^* < 0$) and downstream ($z^* > 0$) medium is given by

$$\frac{\partial}{\partial z^*}[\Gamma(U + v\mu)f_0] = \frac{\partial}{\partial \mu}\left[D_{\mu\mu}(\mu)\frac{\partial f_0}{\partial \mu}\right], \tag{10}$$

i.e.

$$\text{upstream:} \quad \frac{\partial}{\partial z^*}[\Gamma_u(v_1\mu_1 + \beta_u c)f_u] = \frac{\partial}{\partial \mu_1}\left[D_{\mu\mu}(\mu_1)\frac{\partial f_u}{\partial \mu_1}\right],$$

$$\text{downstream:} \quad \frac{\partial}{\partial z^*}[\Gamma_d(v_2\mu_2 + \beta_d c)f_d] = \frac{\partial}{\partial \mu_2}\left[D_{\mu\mu}(\mu_2)\frac{\partial f_d}{\partial \mu_2}\right] \tag{11}$$

2.2. Earlier semi-analytical solutions and present work

In the eigenfunction analysis method (Kirk and Schneider 1987, Kirk et al. 2000) Eqs. (11) for relativistic cosmic rays are solved by expanding $f_{u.d}$ into a complete set of eigenfunctions of the pitch angle scattering operators up- and downstream, and matching the solutions with the continuity condition (Vietri 2003, Blasi and Vietri 2005)

$$f_u(z^* = 0, y_1, \mu_1) = f_d(z^* = 0, y_2, \mu_2) \tag{12}$$

at the shock, and the relations for relativistic cosmic rays

$$y_2 = \Gamma_r y_1(1 + b\mu_1), \quad \mu_2 = \frac{\mu_1 + b}{1 + b\mu_1}, \quad y_1 = \Gamma_r y_2(1 - b\mu_2), \quad \mu_1 = \frac{\mu_2 - b}{1 - b\mu_2}$$

This matching leads to numerical matrix calculations.

Here we follow a different analytical approach aiming for approximate solutions of the up-stream and down-stream Fokker-Planck transport equations (11) using the diffusion approximation. Together with the same (as in earlier semi-analytical work) anisotropic continuity condition (12) we determine the momentum spectrum of accelerated cosmic rays at the shock by integrating Eq. (6) from $z^* = -\eta$ to $z^* = \eta$ and considering the limit $\eta \to 0$ resulting in

$$
\begin{aligned}
& S_d(0, y_2, y_{\|,2}) - S_u(0, y_1, y_{\|,1}) - Q_0(y_1, \mu_1) \\
& + c\Gamma_d \beta_d \lim_{\eta \to 0} \frac{\partial}{\partial y_{\|,2}} [(\beta_d \sqrt{1 + y_2^2} + y_{\|,2})][f_d(\eta, y_2, y_{\|,2}) - f_d(0, y_2, y_{\|,2})] \\
& + c\Gamma_u \beta_u \lim_{\eta \to 0} \frac{\partial}{\partial y_{\|,1}} [(\beta_u \sqrt{1 + y_1^2} + y_{\|,1})][f_u(0, y_1, y_{\|,1}) - f_u(-\eta, y_1, y_{\|,1})] \\
& = \lim_{\eta \to 0} [\int_{-\eta}^{0} dz^* \frac{\partial}{\partial \mu_1} D_{\mu\mu}(\mu_1) \frac{\partial f_u(z^*, y_1, \mu_1)}{\partial \mu_1} + \int_0^{\eta} dz^* \frac{\partial}{\partial \mu_2} D_{\mu\mu}(\mu_2) \frac{\partial f_d(z^*, y_2, \mu_2)}{\partial \mu_2}]
\end{aligned}
\tag{13}
$$

As we demonstrate in the following, with the diffusion approximation applied to the up-stream and down-stream Fokker-Planck transport equations (11), the majority of terms in this lengthly equation will vanish, leaving only

$$
\boxed{S_d(0, y_2, y_{\|,2}) - S_u(0, y_1, y_{\|,1}) = Q_0(y_1, \mu_1)} \tag{14}
$$

with

$$S_d(0, y_2, y_{\|,2}) = \frac{c y_{\|,2} f_d(0, y_2, y_{\|,2})}{\Gamma_d \sqrt{1+y_2^2}} - c\beta_d \Gamma_d (\beta_d \sqrt{1+y_2^2} + y_{\|,2}) \frac{\partial f_d(0, y_2, y_{\|,2})}{\partial y_{\|,2}},$$

$$S_u(0, y_1, y_{\|,1}) = \frac{c y_{\|,1} f_u(0, y_2, y_{\|,1})}{\Gamma_u \sqrt{1+y_1^2}} - c\beta_u \Gamma_u (\beta_u \sqrt{1+y_1^2} + y_{\|,1}) \frac{\partial f_u(0, y_1, y_{\|,1})}{\partial y_{\|,1}} \tag{15}$$

Eqs. (13) and (15) still contain $(y_{\|,1}, y_1)$ and $(y_{\|,2}, y_2)$ which are related to each other using relativistic kinematics.

2.3. Relativistic kinematices

As the flow velocities are constants up- and downstream, the special theory of relativity can be used to calculate the relation between the momentum coordinates up- and downstream. If

$$b = \frac{\beta_u - \beta_d}{1 - \beta_u \beta_d} = \frac{\beta_u (r-1)}{r - \beta_u^2}, \quad \Gamma_r = (1-b^2)^{-1/2} = \Gamma_u \Gamma_d (1 - \beta_u \beta_d) \tag{16}$$

denote the relative velocity of the upstream medium with respect to the downstream medium and the associated relative Lorentz factor, one finds generally with $r = \beta_u / \beta_d$

$$y_2 = \sqrt{\Gamma_r^2(b\mu_1 y_1 + \sqrt{1+y_1^2})^2 - 1}, \quad y_1 = \sqrt{\Gamma_r^2(b\mu_2 y_2 - \sqrt{1+y_2^2})^2 - 1},$$

$$\mu_2 = \frac{y_{\|,2}}{y_2} = \frac{\Gamma_r(y_1\mu_1 + b\sqrt{1+y_1^2})}{\sqrt{\Gamma_r^2(b\mu_1 y_1 + \sqrt{1+y_1^2})^2 - 1}},$$

$$\mu_1 = \frac{y_{\|,1}}{y_1} = \frac{\Gamma_r(y_2\mu_2 - b\sqrt{1+y_2^2})}{\sqrt{\Gamma_r^2(b\mu_2 y_2 - \sqrt{1+y_2^2})^2 - 1}} \tag{17}$$

and with $\beta_1 = y_1/\sqrt{1+y_1^2}$

$$\frac{\partial}{\partial y_{\|,1}} = \frac{1-\mu_1^2}{y_1}\frac{\partial}{\partial \mu_1} + \mu_1\frac{\partial}{\partial y_1}, \quad \frac{\partial}{\partial y_{\|,2}} = \frac{1-\mu_2^2}{y_2}\frac{\partial}{\partial \mu_2} + \mu_2\frac{\partial}{\partial y_2},$$

$$\frac{\partial}{\partial y_2} = \frac{1}{(1+b\beta_1\mu_1)\sqrt{\Gamma_r^2(b\mu_1 y_1 + \sqrt{1+y_1^2})^2 - 1}}\Big[(\beta_1 + b\mu_1)\sqrt{1+y_1^2}\frac{\partial}{\partial y_1} + \frac{b(1-\mu_1^2)}{y_1\sqrt{1+y_1^2}}\frac{\partial}{\partial \mu_1}\Big]$$

$$\frac{\partial \mu_2}{\partial \mu_1} = \frac{\Gamma_r y_1(1 + \frac{b\mu_1}{\beta_1})}{[\Gamma_r^2(b\mu_1 y_1 + \sqrt{1+y_1^2})^2 - 1]^{3/2}} \tag{18}$$

3. Diffusion approximation in the up- and downstream medium

We may write

$$f_0(z^*, y, \mu) = F(z^*, y) + \; g(z^*, y, \mu) \geq 0, \tag{19}$$

with the isotropic part of the cosmic-ray phase space density

$$F(z^*, y) = \frac{1}{2}\int_{-1}^{1} d\mu f_0(z^*, y, \mu), \quad \int_{-1}^{1} d\mu \, g(z^*, y, \mu) = 0 \tag{20}$$

The minimum requirement on $f_0(z^*, y, \mu)$ is that it has no negative values. For small anisotropies $|g(z^*, y, \mu)| \ll F(z^*, p)$ the diffusion approximation (Jokipii 1966, Hasselmann and Wibberenz 1968, RS 1989) provides the diffusion-convection transport equation for the isotropic phase space density $F(z^*, p)$ and the anisotropy $g(z^*, y, \mu)$

$$\frac{\partial}{\partial z}\Gamma[UF - \Gamma\kappa\frac{\partial F}{\partial z^*}] = 0, \quad g(z^*, y, \mu) = \frac{v\Gamma}{4}A(\mu)\frac{\partial F(z^*, y)}{\partial z^*},$$

$$A(\mu) = \int_{-1}^{1} d\mu \, \frac{(1-\mu)(1-\mu^2)}{D_{\mu\mu}(\mu)} - 2\int_{-1}^{\mu} dx \, \frac{(1-x^2)}{D_{\mu\mu}(x)},$$

$$\kappa = -\frac{v^2}{8}\int_{-1}^{1} d\mu \, \mu A(\mu) = \frac{v^2}{8}\int_{-1}^{1} d\mu \, \frac{(1-\mu^2)^2}{D_{\mu\mu}(\mu)} \tag{21}$$

Demanding as spatial boundary conditions

$$F_u(z^* = -\infty, y_1) = 0, \quad F_d(z^* = \infty, y_2) = F_2(y_2),$$
$$F_u(z^* = 0, y_1) \text{ finite}, \; F_d(z^* = 0, y_2) \text{ finite} \tag{22}$$

and spatially-independent diffusion coefficients $\kappa_{1,2}$, the solutions of the up- and downstream diffusion-convection transport equations are then given by

$$F_u(z^* \leq 0, y_1) = F_1(y_1) \exp\left[\frac{\beta_u c z^*}{\Gamma_u \kappa_1}\right], \quad F_d(z^* \geq 0, y_2) = F_2(y_2), \tag{23}$$

implying with Eq. (19) for the approximated up- and downstram anisotropic (but gyrotropic) phase space distribution functions

$$f_u(z^* \leq 0, y_1, \mu_1) \simeq f_u(0, y_1, \mu_1) e^{\frac{\beta_u c z^*}{\Gamma_u \kappa_1}}, \quad f_u(0, y_1, \mu_1) = F_1(y_1) B(\mu_1),$$
$$f_d(z^* \geq 0, y_2, \mu_2) = f_u(0, y_1, \mu_1) \simeq F_2(y_2),$$
$$B(\mu_1) = 1 + \frac{\beta_u \beta_1 c^2}{4\kappa_1} A(\mu_1) = 1 - \frac{2\beta_u}{\beta_1} \frac{A(\mu_1)}{\int_{-1}^{1} d\mu_1 \, \mu_1 A(\mu_1)} \tag{24}$$

Mostly important is the isotropy of the downstream distribution function in its rest frame!

4. Momentum spectrum of accelerated particles at the shock

We will use this essential difference, isotropy of $f_d(0, y_2, \mu_2)$ and anisotropy of $f_u(0, y_1, \mu_1)$, for calculating the downstream and upstream cosmic-ray currents (15). Only when we have accounted for this essential difference in the downstream and upstream cosmic-ray currents, we use the continuity condition (12) at the shock to perform the final averaging over μ_1 to obtain $F_1(p_1)$.[1]
For later use we also calculate the moments

$$\int_{-1}^{1} d\mu\, B(\mu) \left\{ \begin{array}{c} 1 \\ \mu \\ \mu^2 \end{array} \right\} = \left\{ \begin{array}{c} 2 \\ -2\frac{\beta_u}{\beta_1} \\ \frac{2}{3}(1 - \frac{2\beta_u}{\beta_1}\langle\mu\rangle) \end{array} \right\} \tag{25}$$

with

$$\langle\mu\rangle = \frac{\int_{-1}^{1} d\mu \frac{\mu(1-\mu^2)^2}{D_{\mu\mu}(\mu)}}{\int_{-1}^{1} d\mu \frac{(1-\mu^2)^2}{D_{\mu\mu}(\mu)}} \tag{26}$$

[1] It took us some time to realize this important chronological order: first determining the respective currents in their respective up- and downstream system, and then secondly using the continuity condition (12) to average over μ_1. In earlier attemps, we changed the order, using first the continuity condition (13) to calculate the currents (15), and then averaging over μ_1. As a result we could not reproduce the momentum spectra of accelerated cosmic rays at nonrelativistic shocks.

4.1. Proof of cosmic ray jump condition (14)

From Eqs. (24) we find for the different terms occuring in Eq. (13)

$$\lim_{\eta\to 0}[f_u(0.y_1, y_{\|,1}) - f_u(-\eta, y_1, y_{\|,1})] = 0,$$

$$\lim_{\eta\to 0}[f_d(\eta.y_2, y_{\|,2}) - f_u(0, y_2, y_{\|,2})] = 0,$$

$$\lim_{\eta\to 0}\int_{-\eta}^{0} dz^* \frac{\partial}{\partial\mu_1} D_{\mu\mu}(\mu_1)\frac{\partial f_u(z^*, y_1, \mu_1)}{\partial\mu_1}$$

$$= \frac{\Gamma_u\kappa_1 F_1(p_1)}{\beta_u c}\frac{\partial}{\partial\mu_1}D_{\mu\mu}(\mu_1)\frac{\partial}{\partial\mu_1}B(\mu_1)\lim_{\eta\to 0}[1 - e^{-\frac{\beta_u c\eta}{\Gamma_u\kappa_1}}] = 0,$$

$$\lim_{\eta\to 0}\int_{0}^{\eta} dz^* \frac{\partial}{\partial\mu_2} D_{\mu\mu}(\mu_2)\frac{\partial f_d(z^*, y_2, \mu_2)}{\partial\mu_2}$$

$$= F_2(p_2)\frac{\partial}{\partial\mu_1}D_{\mu\mu}(\mu_1)\frac{\partial}{\partial\mu_1}B(\mu_1)\lim_{\eta\to 0}\eta = 0 \qquad (27)$$

Using Eqs. (27), Eq. (13) then reduces to

$$S_d(0, y_2, \mu_2) - S_u(0, y_1, \mu_1) = Q_0(y_1, \mu_1), \qquad (28)$$

proving the earlier stated cosmic ray jump condition (14).

4.2. Transport equation at the shock

In terms of the spherical momentum coordinates (5) and the solutions (24), the up- and downstream cosmic-ray currents (15) at the shock become

$$\frac{S_u(0,y_1,\mu_1)}{c\beta_u\Gamma_u} = [1+\frac{\beta_1\mu_1}{\beta_u}+2\mu_1(\mu_1+\frac{\beta_u}{\beta_1})]F_1(y_1)B(\mu_1)$$
$$-F_1(y_1)(1-\mu_1^2)\frac{\partial}{\partial\mu_1}[(\mu_1+\frac{\beta_u}{\beta_1})B(\mu_1)]$$
$$-\frac{1}{y_1^2}\frac{\partial}{\partial y_1}[y_1^3F_1(y_1)\mu_1(\mu_1+\frac{\beta_u}{\beta_1})B(\mu_1)] \tag{29}$$

and, as $f_d(0,y_2,\mu_2)=F_2(y_2)$ is independent of μ_2,

$$\frac{S_d(0,y_2,\mu_2)}{c\beta_d\Gamma_d} = \mu_2[\mu_2+\frac{\beta_2}{\beta_d}+2(\mu_2+\frac{\beta_d}{\beta_2})]F_2(y_2)$$
$$-\frac{1}{y_2^2}\frac{\partial}{\partial y_2}[y_2^3\mu_2(\mu_2+\frac{\beta_d}{\beta_2})F_2(y_2)] \tag{30}$$

Now, in the last equation we use Eqs. (17) - (18) to obtain for the downstream current

$$S_d(0,y_2,\mu_2) = \frac{cF_2(y_2)}{\Gamma_d}\frac{b+\beta_1\mu_1}{1+b\beta_1\mu_1}$$

$$-c\beta_d\Gamma_r^2\Gamma_d[\beta_d + \frac{b+\beta_1\mu_1}{1+b\beta_1\mu_1}]\frac{(b+\beta_1\mu_1)(1+b\beta_1\mu_1)(1+y_1^2)}{\Gamma_r^2(b\mu_1 y_1+\sqrt{1+y_1^2})^2-1}$$

$$\times\Big(\frac{b(1-\mu_1^2)}{y_1(b\mu_1 y_1+\sqrt{1+y_1^2})}\frac{\partial F_2(y_2)}{\partial\mu_1} + \frac{\beta_1+b\mu_1}{1+b\beta_1\mu_1}\sqrt{1+y_1^2}\frac{\partial F_2(y_2)}{\partial y_1}\Big) \qquad (31)$$

At this stage we use the continuity condition (12) to relate $F_2(y_2)$ to $F_1(y_1)$. Obviously, with the last equation (18)

$$F_2(y_2) = \frac{1}{2}\int_{-1}^{1} d\mu_2\, f_d(0,y_2,\mu_2) = \frac{1}{2}\int_{-1}^{1} d\mu_2\, f_u(0,y_1,\mu_1)$$

$$= \frac{F_1(y_1)}{2}\int_{-1}^{1} d\mu_1\,\frac{\partial\mu_2}{\partial\mu_1}B(\mu_1) = I(b,y_1)F_1(y_1) \qquad (32)$$

with

$$I(b,y_1) = \frac{1}{2\Gamma_r^2}\int_{-1}^{1} d\mu_1\,\frac{B(\mu_1)(1+\frac{b\mu_1}{\beta_1})}{W^{3/2}(b,\mu_1,y_1)},$$

$$W(b,\mu_1,y_1) = (1+\frac{b\mu_1}{\beta_1})^2 + \frac{b^2(1-\mu_1^2)}{y_1^2} = \frac{(1+b\beta_1\mu_1)^2}{\beta_1^2} - \frac{1}{\Gamma_r^2 y_1^2} \qquad (33)$$

Then the downstream current (31) depends also on (y_1,μ_1).

Inserting the currents (29) and (31) the cosmic-ray jump condition (28) then yields

$$\beta_d\Gamma_d F_1(y_1)I(b,y_1)(b+\beta_1\mu_1)\Big(\frac{1}{\beta_d(1+b\beta_1\mu_1)}+\frac{b+\beta_1\mu_1}{\beta_1^2W(b,\mu_1,y_1)}$$
$$+\frac{2[\beta_d(1+b\beta_1\mu_1)+b+\beta_1\mu_1]}{\beta_1^2W(b,\mu_1,y_1)}\Big)$$
$$-\frac{\beta_d\Gamma_d}{(1+b\beta_1\mu_1)y_1^3W^{\frac{3}{2}}(b,\mu_1,y_1)}\Big((\beta_1+b\mu_1)\sqrt{1+y_1^2}\frac{\partial}{\partial y_1}+\frac{b(1-\mu_1^2)}{y_1\sqrt{1+y_1^2}}\frac{\partial}{\partial\mu_1}\Big)$$
$$\big((1+y_1^2)y_1(b+\beta_1\mu_1)W^{1/2}(b,\mu_1,y_1)[b+\beta_1\mu_1+\beta_d(1+b\beta_1\mu_1)]F_1(y_1)I(b,y_1)\big)$$
$$-[1+\frac{\beta_1\mu_1}{\beta_u}+2\mu_1(\mu_1+\frac{\beta_u}{\beta_1})]\beta_u\Gamma_u F_1(y_1)B(\mu_1)$$
$$+\beta_u\Gamma_u F_1(y_1)(1-\mu_1^2)\frac{\partial}{\partial\mu_1}[(\mu_1+\frac{\beta_u}{\beta_1})B(\mu_1)]$$
$$+\frac{\beta_u\Gamma_u}{y_1^2}\frac{\partial}{\partial y_1}[y_1^3F_1(y_1)\mu_1(\mu_1+\frac{\beta_u}{\beta_1})B(\mu_1)]=\frac{Q_0(y_1,\mu_1)}{c} \qquad (34)$$

Eq. (34) is the second important result of our study determining the momentum spectrum $F_1(y_1)$ of accelerated particles at shocks of arbitrary speed for given anisotropy function $B(\mu_1)$. The equation holds for cosmic-ray particles of arbitrary momentum, shock waves of arbitrary speed and and general injection functions $Q_0(y_1,\mu_1)$.

4.3. Pitch-angle averaging

We are particularly interested in the isotropic momentum spectrum $F_1(y_1)$ of accelerated particles at the shock. Therefore we average Eq. (34) over μ_1 , which is equivalent of averaging Eq. (28) over μ_1 leading to

$$\hat{S}_d(0,y_1) - \hat{S}_u(0,y_1) = \frac{\int_{-1}^{1} d\mu_1\, Q_0(y_1,\mu_1)}{2}, \tag{35}$$

where the hat-notation indicates **averaging with respect to μ_1.** In the case of the upstream current of course $\hat{S}_u(0,y_1) = \bar{S}_u(0,y_1)$, whereas for the downstream current in general $\hat{S}_d(0,y_1) \neq \bar{S}_d(0,y_1)$. Eq. (29) provides

$$\frac{\hat{S}_u(0,y_1)}{c\beta_u\Gamma_u} = \frac{F_1(y_1)}{2}\int_{-1}^{1} d\mu_1\,[1+\frac{\beta_1\mu_1}{\beta_u}]B(\mu_1) - \frac{1}{2y_1^2}\frac{\partial}{\partial y_1}[y_1^3 F_1(y_1)\int_{-1}^{1} d\mu_1\,\mu_1$$
$$\times(\mu_1+\frac{\beta_u}{\beta_1})B(\mu_1)] = -\frac{1}{3y_1^2}\frac{d}{dy_1}\big(y_1^3 F_1(y_1)[1-2\frac{\beta_u}{\beta_1}\langle\mu_1\rangle - 3\frac{\beta_u^2}{\beta_1^2}]\big), \tag{36}$$

where we calculated with the moments (25)

$$\int_{-1}^{1} d\mu_1\,[1+\frac{\beta_1\mu_1}{\beta_u}]B(\mu_1) = 2+\frac{\beta_1}{\beta_u}(-\frac{2\beta_u}{\beta_1}) = 0,$$
$$\int_{-1}^{1} d\mu_1\,[\mu_1+\frac{\beta_u}{\beta_1}]\mu_1 B(\mu_1) = \frac{2}{3}[1-2\frac{\beta_u}{\beta_1}\langle\mu_1\rangle - 3\frac{\beta_u^2}{\beta_1^2}] \tag{37}$$

Likewise, after straightforward but tedious algebra

$$\frac{\hat{S}_d(0,y_1)}{c\beta_d\Gamma_d} = F_1(y_1)I(b,y_1)\Big(\frac{1}{\beta_d b}[1-\frac{\text{artanh}\, b\beta_1}{\Gamma_r^2 b\beta_1}]$$
$$+\frac{3}{2\beta_1^2}\int_{-1}^{1} d\mu_1\,\frac{G}{W} - \frac{b}{y_1^2}\int_{-1}^{1} d\mu_1\,\frac{(b+\beta_1\mu_1)G}{(1+b\beta_1\mu_1)^2 W}$$
$$-\frac{\beta_d}{2\beta_1^2}\int_{-1}^{1} d\mu_1\,\frac{(b+\beta_1\mu_1)(1+b\beta_1\mu_1)}{W}\Big)$$
$$-\frac{1}{2y_1^2}\frac{\partial}{\partial y_1}\Big(\frac{y_1^3 F_1(y_1)I(b,y_1)}{\beta_1^2}\int_{-1}^{1} d\mu_1\,\frac{(1+\frac{b\mu_1}{\beta_1})G}{(1+b\beta_1\mu_1)W}\Big) \tag{38}$$

with

$$G(b,\mu_1,y_1) = (b+\beta_1\mu_1)[b+\beta_1\mu_1+\beta_d(1+b\beta_1\mu_1)], \tag{39}$$

the pitch-angle averaged transport eqiation (35) then reads

$$\boxed{\Omega(b,y_1)F_1(y_1) + \frac{1}{y_1^2}\frac{d}{dy_1}(y_1^3 F_1(y_1)T(b,y_1)) = \frac{\int_{-1}^{1} d\mu_1\, Q_0(y_1,\mu_1)}{2}}, \tag{40}$$

where we introduce the convection rate

$$\frac{\Omega(b,y_1)}{c\beta_d\Gamma_d I(b,y_1)} = \frac{1}{\beta_d b}[1 - \frac{\text{artanh}\, b\beta_1}{\Gamma_r^2 b\beta_1}] + \frac{3}{2\beta_1^2}\int_{-1}^{1} d\mu_1 \frac{G}{W}$$
$$-\frac{b}{2y_1^2}\int_{-1}^{1} d\mu_1 \frac{(b+\beta_1\mu_1)G}{(1+b\beta_1\mu_1)^2 W}$$
$$-\frac{\beta_d}{2\beta_1^2}\int_{-1}^{1} d\mu_1 \frac{(b+\beta_1\mu_1)(1+b\beta_1\mu_1)}{W} \tag{41}$$

and the acceleration rate

$$T(b,y_1) = \frac{c\beta_u\Gamma_u}{3}[1 - 2\frac{\beta_u}{\beta_1}\langle\mu_1\rangle - 3\frac{\beta_u^2}{\beta_1^2}] - \frac{c\beta_d\Gamma_d I(b,y_1)V(b,y_1)}{3},$$
$$V(b,y_1) = \frac{3}{2}[\int_{-1}^{1} d\mu_1 \frac{(1+\frac{b\mu_1}{\beta_1})(\mu_1+\frac{b}{\beta_1})^2}{(1+b\beta_1\mu_1)W} + \frac{\beta_d}{\beta_1}\int_{-1}^{1} d\mu_1 \frac{(1+\frac{b\mu_1}{\beta_1})(\mu_1+\frac{b}{\beta_1})}{W}] \tag{42}$$

The general transport equation (40) together with the convection rate (41) and the acceleration rate (42) are our third important result. These equations hold for arbitrary shock speeds, arbitrary cosmic-ray momenta and general injection functions at the shock.

4.4. Solution of the transport equation at the shock

Eq. (40) for general injection rates $Q_0(p_1, \mu_1)$ is solved by

$$F_1(y_1) = \frac{1}{2y_1^3 T(b, y_1)} \int_0^{y_1} dy\, y^2 \exp[-\int_y^{y_1} dy' \, \frac{\Omega(b, y')}{y' T(b, y')}] \int_{-1}^{1} d\mu_1 \, Q_0(y, \mu_1) \tag{43}$$

as general solution valid for arbitrary cosmic-ray particle momenta and any particle injection rate at the shock.

We adopt as illustrative example the isotropic monomomentum injection rate $Q_0(y_1, \mu_1) = Q_1 \delta(y_1 - y_0)$. We then obtain for the momentum spectrum of accelerated cosmic rays at the shock

$$F_1(y_1 \geq y_0) = \frac{3Q_1 y_0^2}{y_1^3 T(y_1, b)} \exp[-\int_{y_0}^{y_1} dy \, \frac{\Omega(b, y)}{y T(b, y)}], \tag{44}$$

which is finite as $y_1 \to 0$ and $y_1 \to \infty$, respectively.

For the (in general momentum-dependent) power-law spectral index

$$F_1(y_1) = A_0 \left(\frac{y_1}{y_0}\right)^{-q(r,y_1)} \quad , \quad q(r,y_1) = -\frac{d\ln F_1(y_1)}{d\ln(y_1/y_0)} \tag{45}$$

we obtain

$$q(r,y_1) = 3 + y_1 \frac{d\ln T(b,y_1)}{dy_1} + \frac{\Omega(b,y_1)}{T(b,y_1)} \tag{46}$$

The power-law spectral index (46) depends on the ratio of the convection rate $\Omega(y_1,b)$ and the acceleration rate $T(y_1,b)$. The solutions (43) - (46) are the fourth important results of our study.

5. Extreme nonrelativistic shock speeds

We first consider the straightforward limit of extremely nonrelativistic shock speeds with $b = 0$ and $\Gamma_r = 1$ which serves as testing result for our later nonrelativistic expansion of finite but small $b \ll 1$. This case is somewhat unphysical as $b \simeq \beta_u - \beta_d = 0$ implies $\beta_u = \beta_d$, so that no nonrelativistic shock occurs.

With $W(b = 0, \mu_1, y_1) = 1$, $G(y_1, \mu_1, b) = \beta_1^2\mu_1^2 + \beta_d\beta_1\mu_1$ and $I(b = 0, y_1) = 1$ the averaged downstream cosmic-ray current in this limit with $\Gamma_d \simeq 1$ reduces to[2]

$$\left[\frac{\hat{S}_d(0, y_1)}{c\beta_d}\right]_{b=0} = \frac{F_1(y_1)}{2}\left([\frac{\beta_1}{\beta_d} + \frac{2\beta_d}{\beta_1}]\int_{-1}^{1} d\mu_1\, \mu_1 + 3\int_{-1}^{1} d\mu_1\, \mu_1^2\right)$$
$$-\frac{1}{2y_1^2}\frac{\partial}{\partial y_1}[\beta_1 y_1(1 + y_1^2)F_1(y_1)\int_{-1}^{1} d\mu_1\, \mu_1(\beta_d + \beta_1\mu_1)]$$
$$= F_1(y_1) - \frac{1}{3y_1^2}\frac{\partial}{\partial y_1}[y_1^3 F_1(y_1)] \qquad (47)$$

[2] It is important to realize that the averaging of the dominating term $\propto (\beta_1/\beta_d)$ for nonrelativistic shock speeds with $\beta_d \ll 1$ vanishes. This is not the case if we account for the difference between μ_1 and μ_2 if $b \neq 0$.

With the averaged upstream cosmic-ray current (36) taken in the limit $\Gamma_u \simeq 1$ we find for the transport equation (40) for extremely nonrelativistic shock speeds

$$\beta_d F_1(y_1) + \frac{1}{3y_1^2}\frac{d}{dy_1^2}\left(y_1^3 F_1(y_1)\left[\beta_u[1-2\frac{\beta_u}{\beta_1}\langle\mu_1\rangle - 3\frac{\beta_u^2}{\beta_1^2}] - \beta_d\right]\right) = \frac{1}{2c}\int_{-1}^{1} d\mu_1\, Q_0(y_1,\mu_1) \tag{48}$$

For relativistic cosmic rays $\beta_1 \simeq 1$ and $\beta_u = r\beta_d \ll 1$ the transport equation (48) provides

$$\beta_d F_1(y_1) + \frac{\beta_u - \beta_d}{3y_1^2}\frac{d(y_1^3 F_1(y_1))}{dy_1^2} = \frac{1}{2c}\int_{-1}^{1} d\mu_1\, Q_0(y_1,\mu_1), \tag{49}$$

agreeing exactly with the transport equation used in nonrelativistic acceleration theory.

However, it strictly holds only in the limit $b = 0$, corresponding to $\beta_u = \beta_d$, so that Eq. (49) becomes

$$\beta_d F_1(y_1) = \frac{1}{2c}\int_{-1}^{1} d\mu_1\, Q_0(y_1,\mu_1) \tag{50}$$

with the correct solution

$$F_1(y_1) = \frac{1}{2\beta_d c}\int_{-1}^{1} d\mu_1\, Q_0(y_1,\mu_1) \tag{51}$$

6. Correct limit of nonrelativistic shocks

The correct limit of nonrelativistic shocks with small but finite values of $b = (r-1)\beta_d \ll 1$, implying $b/\beta_d = r-1$, yields with the approximation

$$\frac{1}{W(\mu_1)} \simeq \frac{1}{W_0(\mu_1)} - \frac{b^2(1-\mu_1^2)}{y_1^2 W_0^2(\mu_1)} \quad W_0(\mu_1) = (1+\frac{b\mu_1}{\beta_1})^2 \tag{52}$$

to second order in b for the convection rate

$$\Omega(b \ll 1, y_1) \simeq c\beta_d I(b \ll 1, y_1)[1+\frac{b}{\beta_d}(1-\frac{\beta_1^2}{3})] = c\beta_d I(b \ll 1, y_1)[1+(r-1)(1-\frac{\beta_1^2}{3})]$$

$$\simeq c\beta_d[r - \frac{r-1}{3}\beta_1^2 + \frac{\beta_d^2 r(r-1)(7r-1)}{3\beta_1^2}] \tag{53}$$

for symmetric ($D_1(-\mu_1) = D_1(\mu_1)$) upstream Fokker-Planck coefficients, where according to Eq. (26) the first moment vanishes $\langle \mu_1 \rangle = 0$.
Likewise, the correct nonrelativistic acceleration rate becomes

$$T(b \ll 1, y_1) \simeq \frac{c\beta_d r}{3}\Big(1 - 3\frac{r^2\beta_d^2}{\beta_1^2}$$

$$- \frac{1}{r}[1 + \frac{(r-1)(7r-1)\beta_d^2}{3\beta_1^2}][1 + \frac{2(3r+2)(r-1)\beta_d^2}{5\beta_1^2}]\Big) \tag{54}$$

6.1. Injection threshold

Because the convection rate (53) is always positive, cosmic-ray particles only are accelerated for a positive acceleration rate (54) leading to the condition for the cosmic-ray velocities

$$\beta_1 > \beta_c(r) = \frac{\beta_u \sqrt{u_c(r)}}{r}, \quad u_c(r) = \frac{45r^3 + (r-1)(53r+7)}{30(r-1)}[1 + \sqrt{1 + \frac{120(r-1)^3(3r+2)(7r-1)}{[45r^3 + (r-1)(53r+7)]^2}}] \tag{55}$$

$\beta_c(r)$ which is much smaller than unity so that the injection velocity (55) corresponds to cosmic-ray momenta $y_1 > y_c \simeq \beta_c$.

The injection threshold (55) is well approximated (see Fig. 2) by

$$\beta_c(r) \simeq \sqrt{3}\beta_u \frac{1 + \sqrt{r-1}}{\sqrt{r-1}} \tag{56}$$

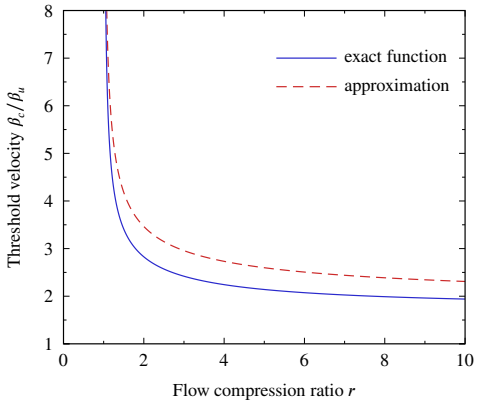

Figure 2: Injection threshold velocity β_c/β_u as a function of the flow compression ratio r for a nonrelativistic shock. The blue full curve shows the exact function (55) and the red dashed curve the approximation (56).

This settles the injection threshold for nonrelativistic shock acceleration. Only cosmic-ray particles with momenta or velocities $y_1 \simeq \beta_1 > \beta_C$ are accelerated by nonrelativistic shocks.

For all flow compression ratios the injection threshold is always larger than $\sqrt{3}\beta_u$. This minimum value is approached for large compression ratios $r \gg 1$. For shocks with compression ratios close to unity, the injection threshold increases $\propto (r-1)^{-1/2}$.

Fraction of thermal particles subject to diffusive shock acceleration

$$\frac{n_a}{n_0} \simeq \frac{2\sqrt{3}M_{\rm th}}{\sqrt{\pi}} e^{-3M_{\rm th}^2} \tag{57}$$

and fraction of thermal energy density particles subject to diffusive shock acceleration

$$\frac{E_a}{E_0} \simeq \frac{4\sqrt{3}M_{\rm th}^3}{\sqrt{\pi}} e^{-3M_{\rm th}^2} \tag{58}$$

depend on thermal Mach number $M_{\rm th} = \beta_u c / v_{\rm th}$.

For $M_{\rm th} = 5$

$$\frac{n_a}{n_0} \simeq 2.7 \cdot 10^{-33}, \quad \frac{E_a}{E_0} \simeq 1.3 \cdot 10^{-30} \tag{59}$$

6.2. Momentum spectrum of accelerated cosmic-ray particles at the shock

With the acceleration and convection rates (53) and (54) the power-law spectral index (46) becomes

$$q(r, y_1 \geq y_0 > y_c) = 3 + y_1 \frac{d}{dy_1} \ln[1 - \frac{y_c^2}{\beta_1^2}] + \frac{3r}{r-1} \frac{1 - \frac{r-1}{3r}\beta_1^2 + \frac{r(r-1)^2(7r-1)y_c^2}{9(r^3-r^2+1)\beta_1^2}}{1 - \frac{y_c^2}{\beta_1^2}}$$

$$= 3 + \frac{2y_c^2\beta_1^2}{y_1^2(\beta_1^2 - y_c^2)} + \frac{3r}{r-1} \frac{\beta_1^2 - \frac{r-1}{3r}\beta_1^4 + \frac{r(r-1)^2(7r-1)y_c^2}{9(r^3-r^2+1)}}{\beta_1^2 - y_c^2}, \qquad (60)$$

which is shown in Fig. 3 for $\beta_1 > 1.1 y_c$ and different flow compression ratios. For relativistic cosmic rays with $\beta_1 \simeq 1 \gg y_c$ the spectral index (60) approaches

$$q(r, y_1 > 1) = q(r) \simeq \frac{5r-2}{r-1} = 2 + \frac{3r}{r-1} \qquad (61)$$

This spectral index for relativistic cosmic rays differs from the classical nonrelativistic shock acceleration theory spectral index $3r/(r-1)$ by the additional factor 2.

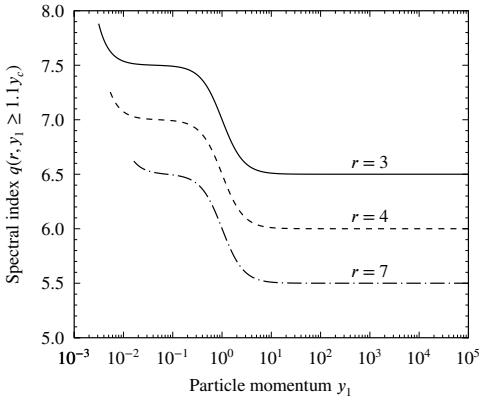

Figure 3: Power-law spectral index (60) for a nonrelativistic shock with $\beta_d = 10^{-4}$ as a function of particle momentum $y_1 = 1.1 y_c(r)$ for three different flow compression ratios $r = 3$ (full curve), $r = 4$ (dashed curve) and $r = 7$ (dot-dashed curve). Note that $y_c(r)$ depends on r.

For nonrelativistic cosmic rays with $\beta_1 \simeq y_1 \gg y_c$ the spectral index (60) approaches

$$q(r, y_c \ll y_1 < 1) = q_0(r) \simeq= \frac{3(2r-1)}{r-1} = 1 + q(r, y_1 > 1) \qquad (62)$$

This spectral index for nonrelativistic cosmic rays differs from the classical nonrelativistic shock acceleration theory spectral index $3r/(r-1)$ by the additional factor 3. Our theory predics a power-law flattening by unity in the spectral index value at the transition from nonrelativistic to relativistic cosmic-ray momenta.

Evidently, our strictly relativistic shock acceleration theory, accounting for the different momentum coordinates in the up- and downstream media, where the diffusion approximation has been applied, implies a much weaker efficiency for the acceleration of relativistic cosmic rays at nonrelativistic shocks, when the nonrelativistic limit of small but finite shock speeds in the relativistic theory is considered.[3]

[3] This is not the first time in physics, that the nonrelativistic limit of a relativistic theory yields different result than the nonrelativistic theory: the most famous example is the nonrelativistic limit of the Dirac equation in quantum mechanics which differs in many aspects from the nonrelativistic Schrödinger equation.

7. Summary and conclusions

- The analytical theory of diffusive cosmic ray acceleration at parallel stationary shock waves of arbitrary speed has been developed. Starting from the Fokker-Planck particle transport equation in the mixed comoving coordinate system we derived for the first time the correct cosmic-ray jump conditions at the shock relating the upstream and downstream anisotropic cosmic-ray currents.

- The anisotropic upstream and downstream cosmic-ray currents are calculated from a diffusion approximation of particle transport in the upstream and downstream medium.

- Pitch-angle averaging the cosmic-ray current jump condition provides a general solution for the isotropic momentum spectrum of accelerated particles at the shock valid for arbitrary shock speeds, arbitrary cosmic-ray momenta and general injection functions at the shock, determined by the ratio of general acceleration and convection rates.

- The correctly taken limit for nonrelativistic shocks and relativistic cosmic-ray momenta leads to the power-law momentum spectrum $F_1(y_1) \propto y_1^{-q(r)}$, where the power-law spectral index q_0 is a factor 2 greater than the standard spectral index from nonrelativistic shock acceleration theory.

- For the first time we settled the injection problem for nonrelativistic shocks. Only cosmic rays with velocities $\beta_1 > \beta_c(r) \simeq \sqrt{3}\beta_u \frac{1+\sqrt{r-1}}{\sqrt{r-1}} \geq \sqrt{3}\beta_u$ are accelerated by nonrelativistic shocks.

- Our strictly relativistic shock acceleration theory, accounting for the different momentum coordinates in the up- and downstream media, where the diffusion approximation has been applied, implies a much weaker efficiency for the acceleration of relativistic cosmic rays at nonrelativistic shocks, when the nonrelativistic limit of small but finite shock speeds in the relativistic theory is considered.

- This does not necessarily imply that nonrelativitic shock waves are in general inadequate to accelerate galactic cosmic rays at supernova remnants: we only demonstrated that shock waves with solely magnetostatic turbulence are inadequate to do so. Additional second-order Fermi acceleration processes, resulting from MHD plasma turbulence with finite non-zero phase speeds and cross helicities, may lead to more efficient acceleration and flatter particle momentum spectra (see e.g. Vainio and Schlickeiser 1999; Schlickeiser, Lerche and Campeanu 1993).

- We are busy investigating the other important limit of our general theory namely cosmic rays accelerated at relativistic shocks.

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