Physics of Hydrodynamic Turbulence and Passive Scalar Transfer

6th Korean Astrophysics Workshop on
Fundamental Process of Astrophysical Turbulence

Asia Pacific Center for Theoretical Physics (APCTP)
Pohang, Korea
16 -- 19 November, 2011

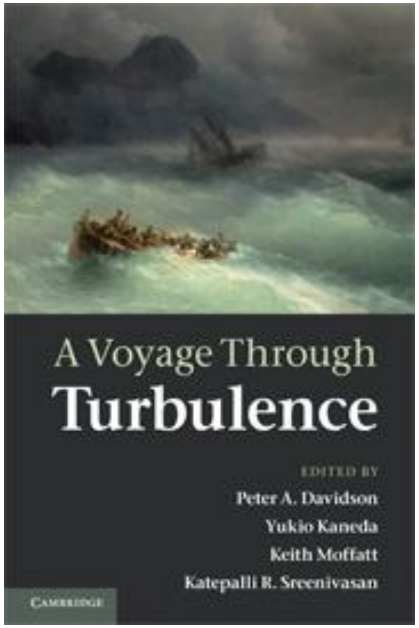
Toshiyuki Gotoh

Nagoya Institute of Technology

Kolmogorov's turbulence theory (1941)

Milestone of turbulence theory

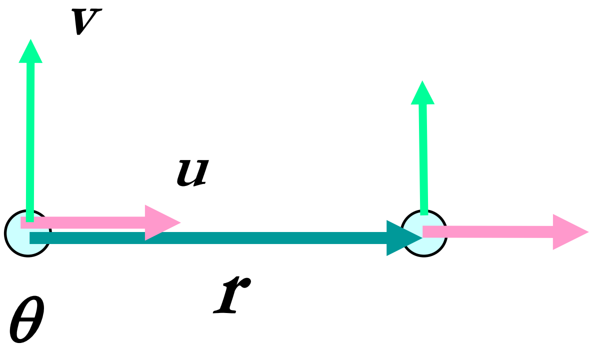
Prandtl, Onsager, von Weizsaeker, Heisenberg



Cambridge Univ. Press 2011

Velocity and scalar increments

$$U = u(x+r) - u(x)$$
$$V = v(x+r) - v(x)$$
$$\Theta = \theta(x+r) - \theta(x)$$



Two hypotheses and dimensional analysis

1. At large R and small r, turbulent field is locally homogeneous and isotropic, and the probability distribution of U and V are uniquely determined by

$$\bar{\varepsilon} \quad \text{finite} \quad \text{as} \quad \nu \longrightarrow 0$$

$\bar{\varepsilon}$ — Average energy dissipation rate

ν — Viscosity

⇒ $\eta = (\nu^3/\bar{\varepsilon})^{1/4}$ — Kolmogorov length

the smallest scale

2. When $\eta \ll r$, the probability distribution is independent of the viscosity

Inertial Range $\eta \ll r \ll L$

PDF $$(\bar{\epsilon} r)^{2/3} P(U,V,r) = \bar{P}(U/(\bar{\epsilon} r)^{1/3}, V/(\bar{\epsilon} r)^{1/3})$$

(Probability Density Function)

Moment $$\langle U^p V^q \rangle \propto r^{\zeta_{p,q}}, \quad \zeta_{p,q} = \frac{p+q}{3}$$

Normal scaling

$$\left\langle U^2 \right\rangle = C_2 \bar{\epsilon}^{2/3} r^{2/3}$$

$$E(k) = K \bar{\epsilon}^{2/3} k^{-5/3}$$ ***Kolmogorov's Spectrum***

$$\left\langle U^3 \right\rangle = -\frac{4}{5} \bar{\epsilon} r$$

Kolmogorov's 4/5 Law
(Onsager 1945)
(asymptotically exact)

Characteristic time $$\tau_r = \frac{r}{(\bar{\epsilon} r)^{1/3}} = \bar{\epsilon}^{-1/3} r^{2/3}$$

DNS of HIT (Homogeneous Isotropic Turbulence) and passive scalar

Periodic boundary condition (homogeneity)

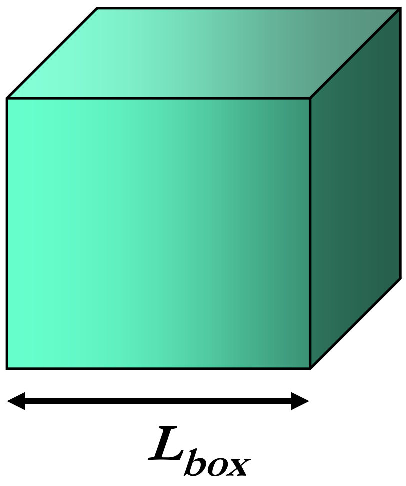
$$u(x,t) = \sum_{|k|\le K_{max}} u(k,t)e^{ik\cdot x}, \quad u(k,t) = \frac{1}{L_{box}^3}\int_{L_{box}^3} u(x,t)e^{-ik\cdot x}dx$$

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) u(k,t) = M(k) : \sum_{k=p+q} u(p,t)u(q,t) + f(k,t)$$

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) \theta(k,t) = ik \cdot \sum_{k=p+q} u(p,t)\theta(q,t) + f_\theta(k,t)$$

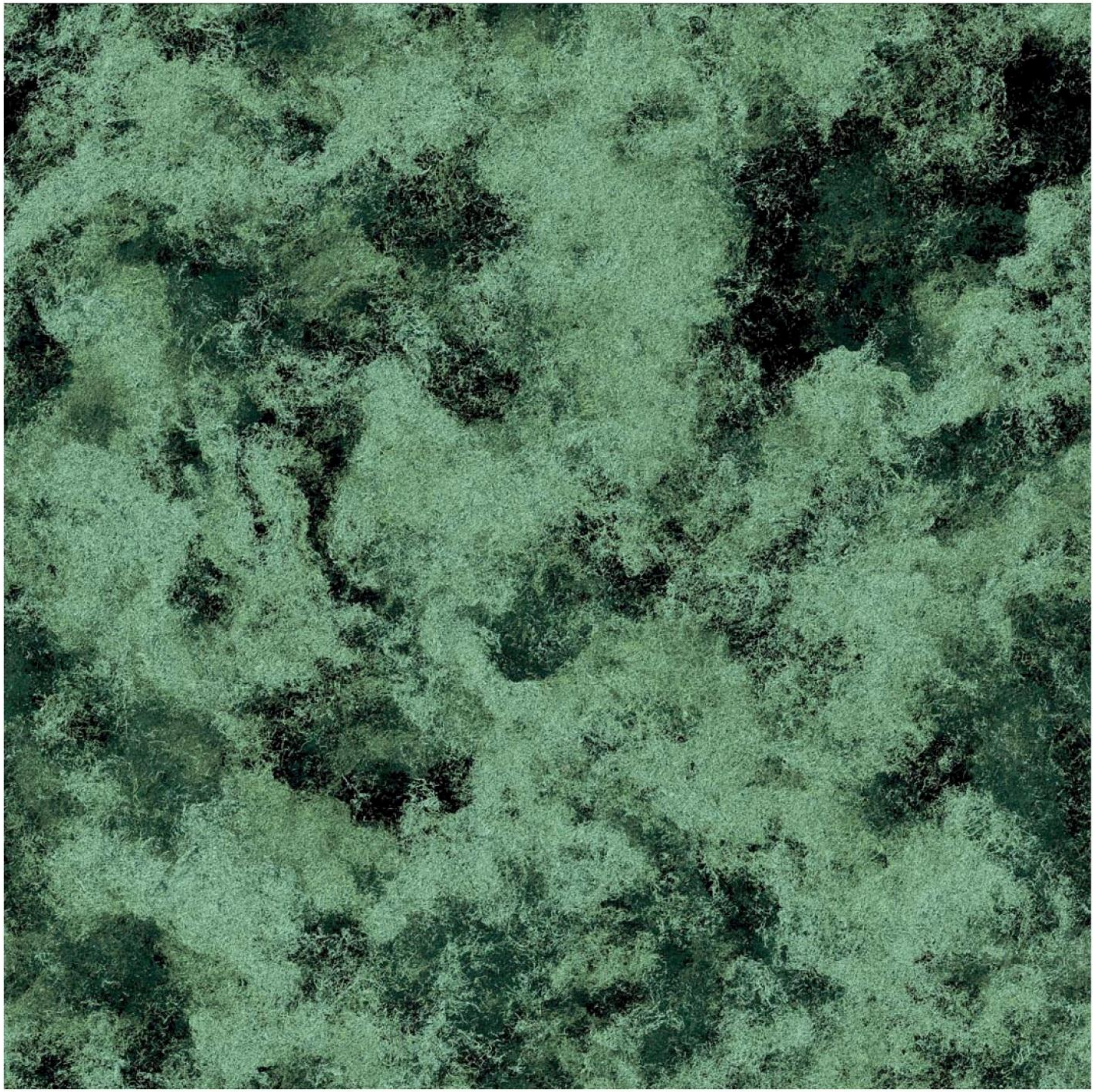
$$M_{lmn}(k) = \frac{i}{2}\left(k_m P_{ln}(k) + k_n P_{lm}(k)\right),$$

$$P_{ij}(k) = \delta_{ij} - \frac{k_i k_j}{k^2}$$

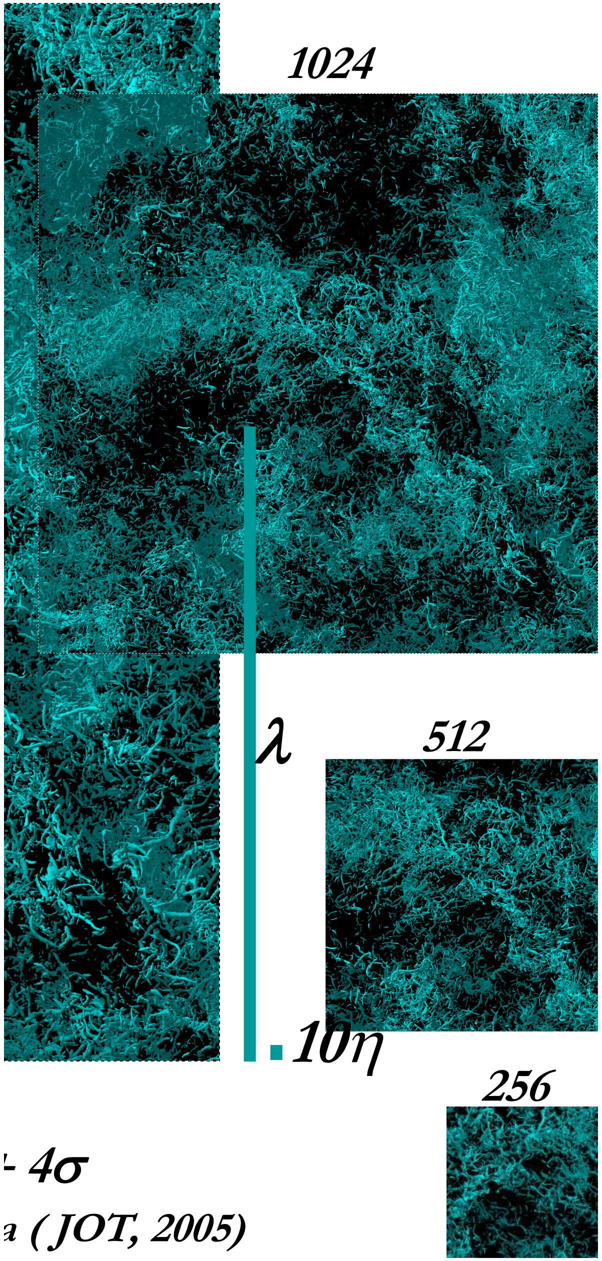


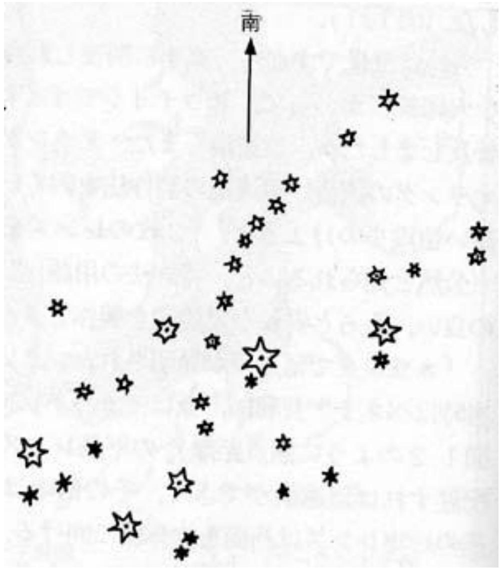
Exact sol. of Poisson eq. for pressure
High accuracy
Efficient use of computer resources in spatial resolution

HIT DNS

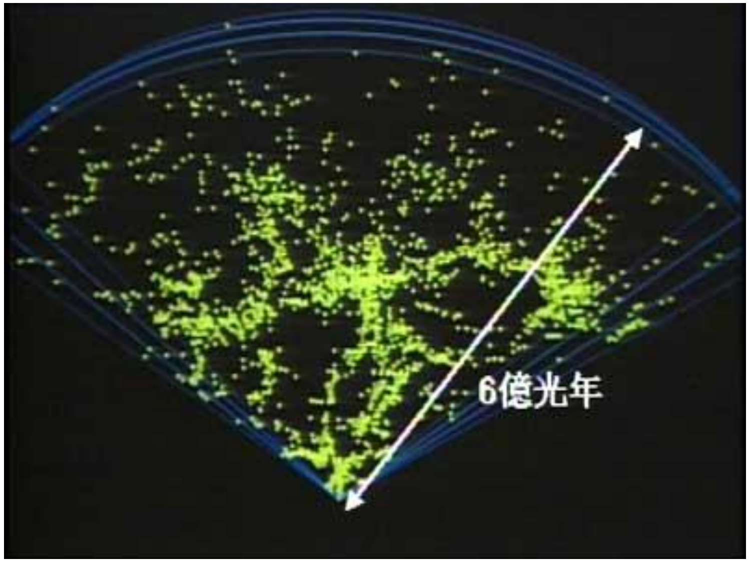


Ishihara, Gotoh、Kaneda Ann.Rev.Fluid Mech.(2010)





Sketch of Pleiades cluster by Galileo (1610)



Great wall in the universe
de Lapparent et al. (1986)
http://www.nhk.or.jp/school/junior/yougo26.html#010

Galileo's telescope (1610)

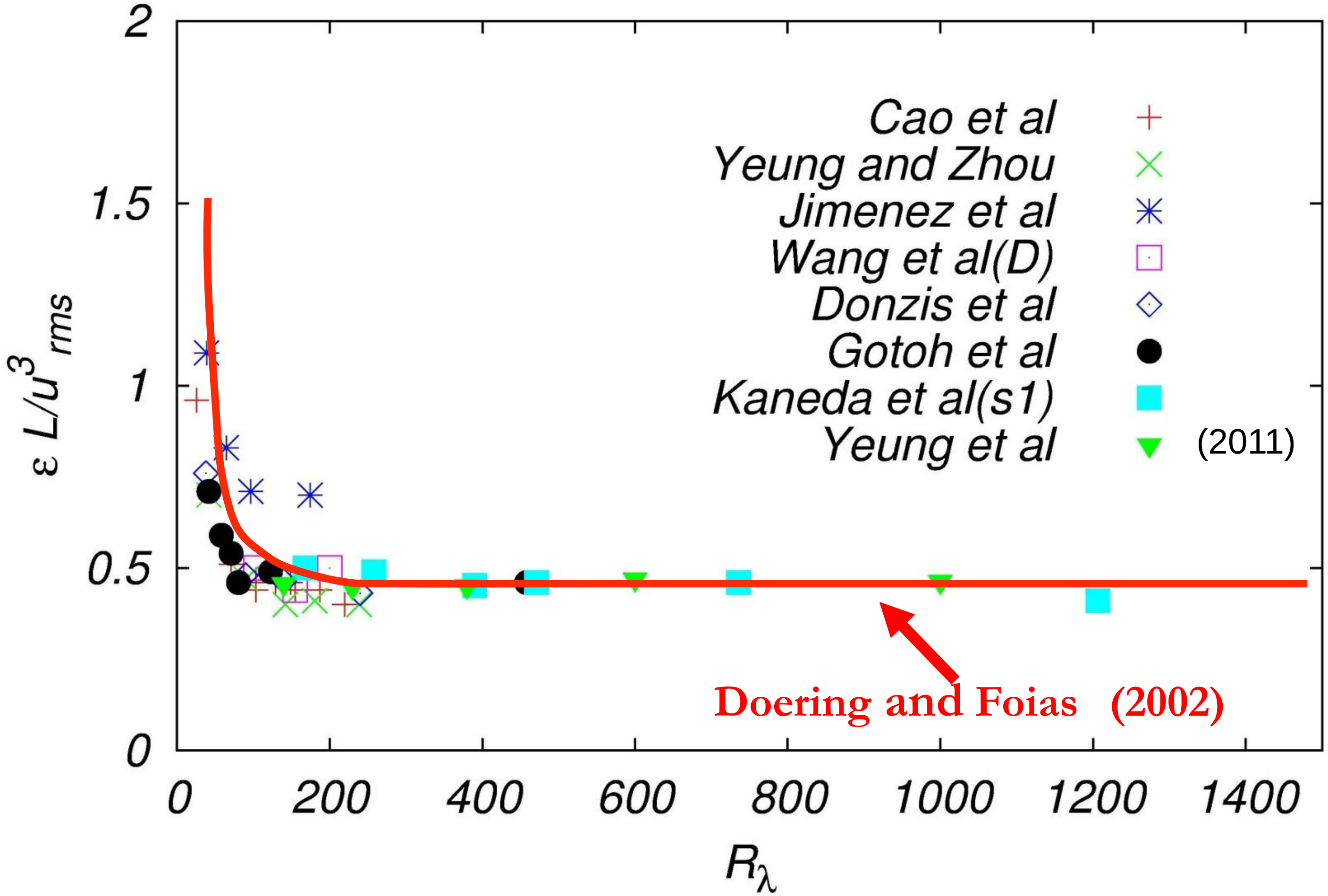
Subaru Telescope (1999)

Normalized Energy Dissipation

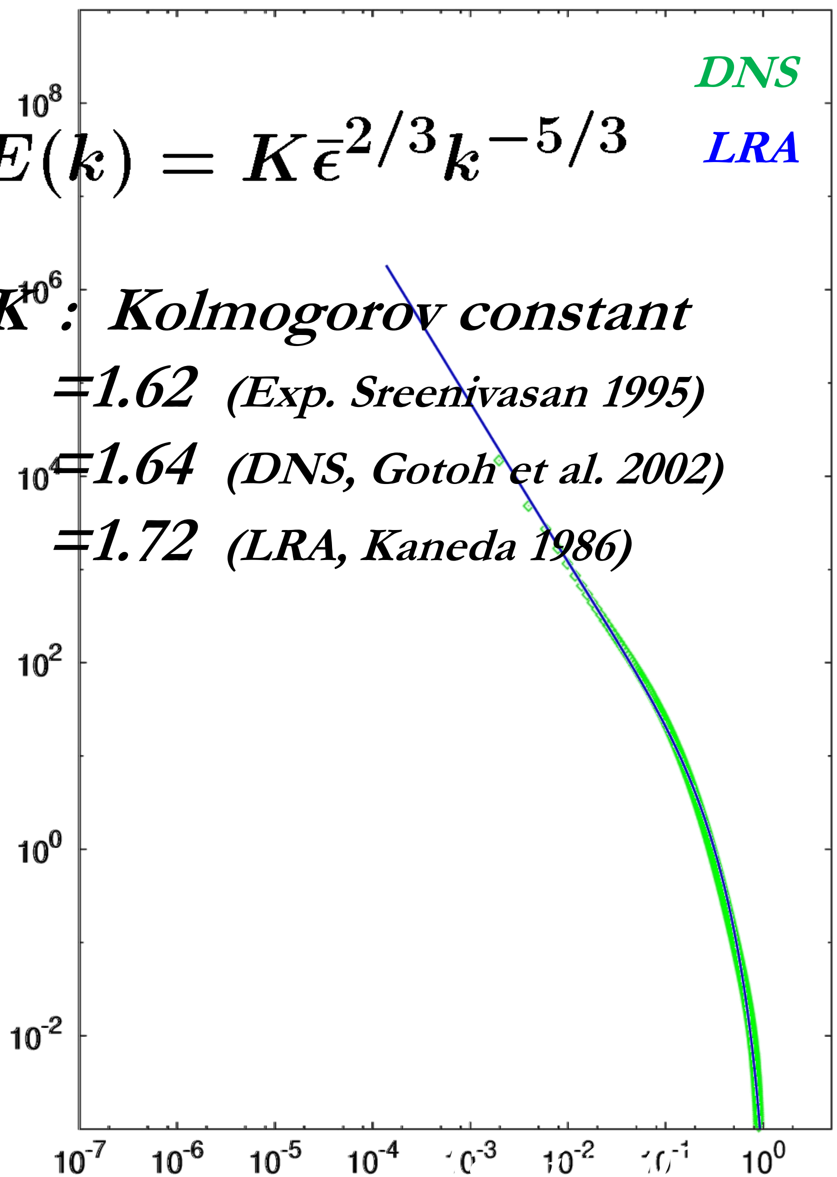
$$\beta = \frac{\varepsilon L}{u_{rms}^3}$$

$$\beta \le a\left(1 + \sqrt{1 + (b/R_\lambda)^2}\right), \quad a = 0.2,\ b = 90$$

Doering and Foias (2002) *Donzis et al. (2005)*



Kolmogorov Spectrum



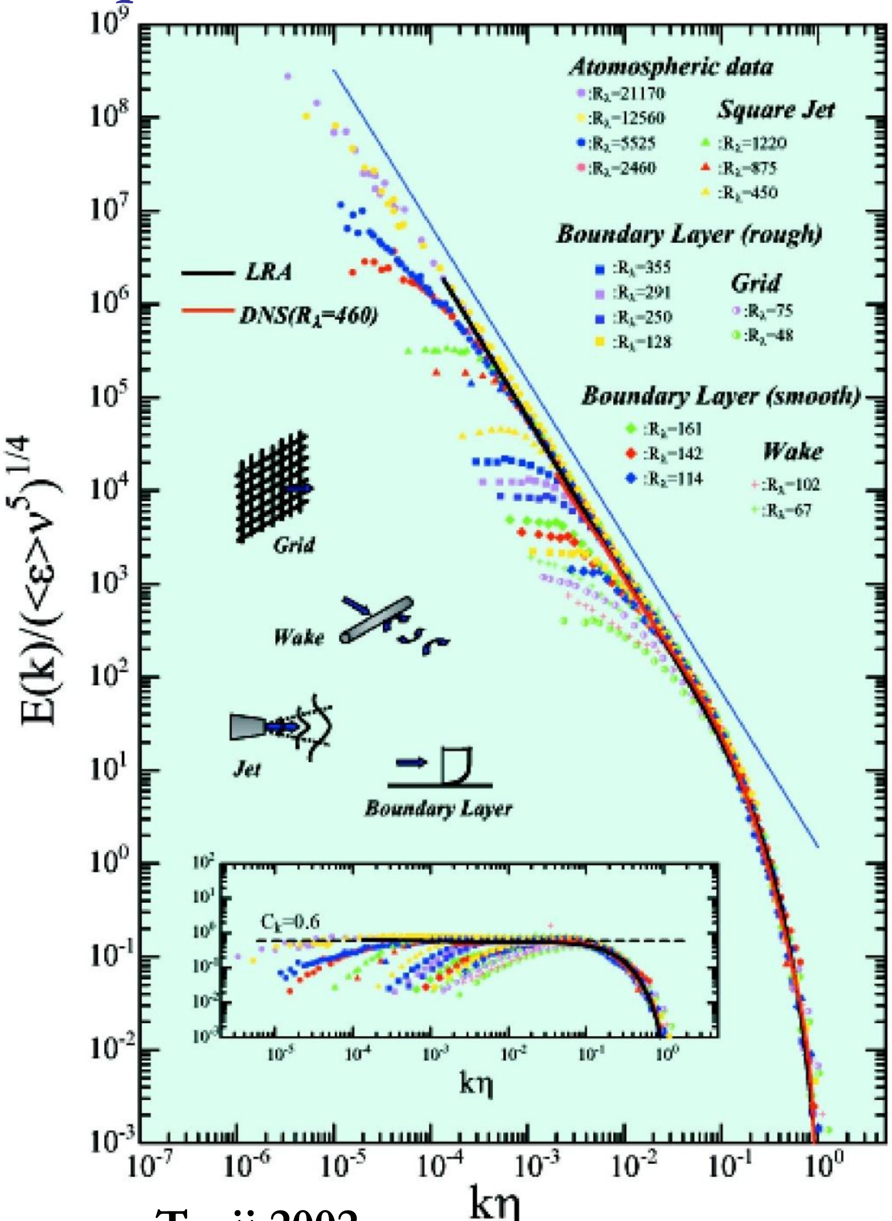
$$E(k) = K\bar{\epsilon}^{2/3}k^{-5/3}$$

K: *Kolmogorov constant*

=1.62 (Exp. Sreenivasan 1995)

=1.64 (DNS, Gotoh et al. 2002)

=1.72 (LRA, Kaneda 1986)



Tsuji 2002

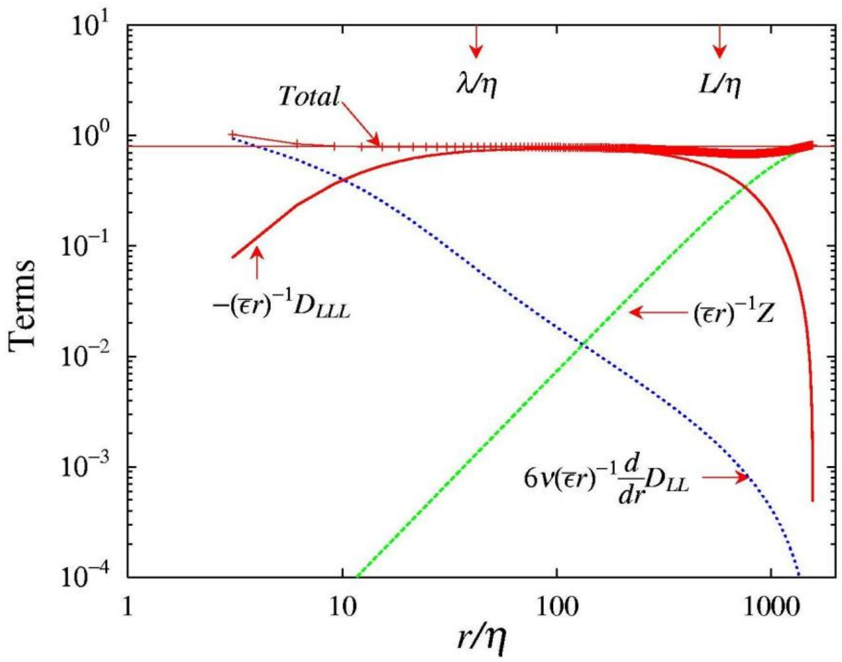
Energy budget in scale

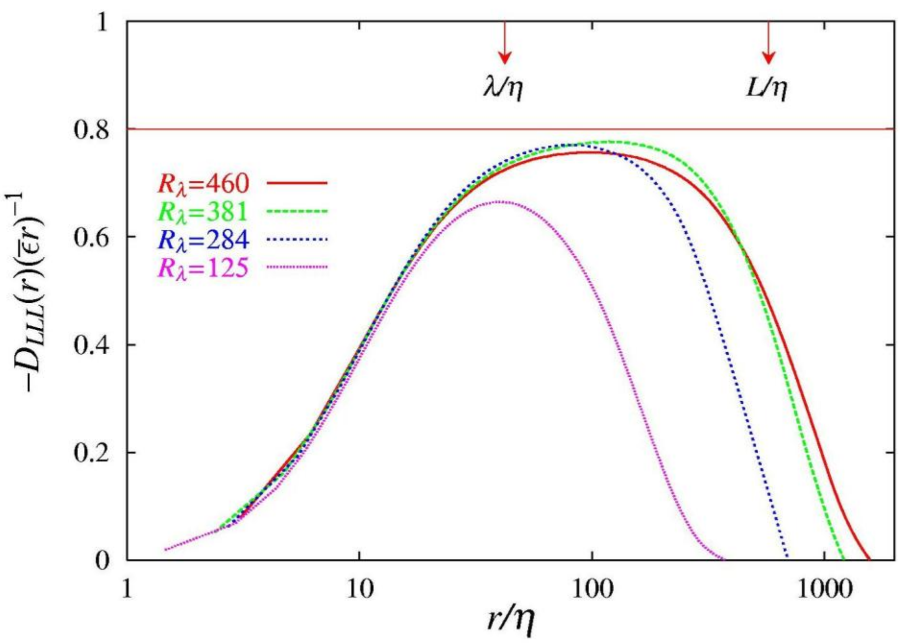
$$D_{LL}(r) = \langle U^2 \rangle \quad D_{LLL}(r) = \langle U^3 \rangle$$

KHK equation

$$\frac{4}{5}\bar{\epsilon}r = -D_{LLL} + 6\nu\frac{dD_{LL}}{dr} + Z_{force}(r)$$

4/5 law





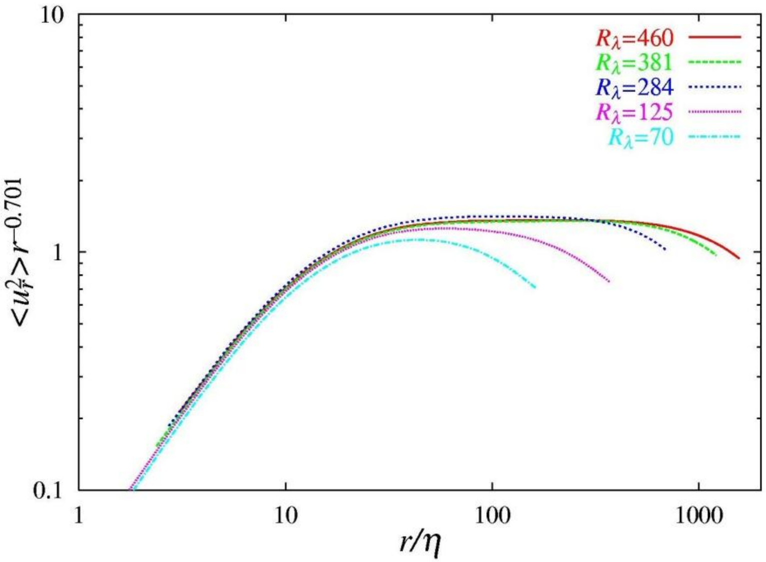
(Gotoh et al. 2002)

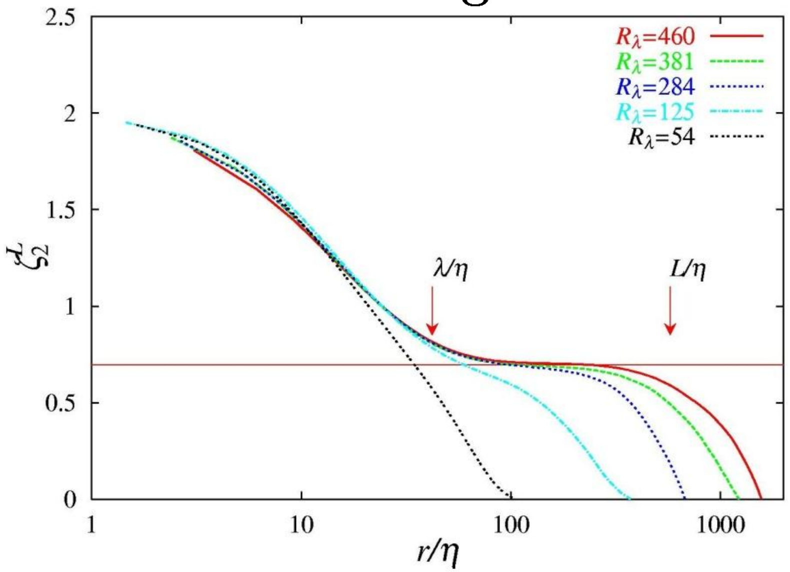
- ***KHK equation is satisfied***
- ***Slow approach to 4/5 law with increase of R_λ***

Intermittency deviation from Kolmogorov (K41) theory

Structure functions of velocity increments

$$r^{-0.701}\left\langle U^2\right\rangle \qquad\qquad \frac{d\log\left\langle U^2\right\rangle}{d\log r}$$



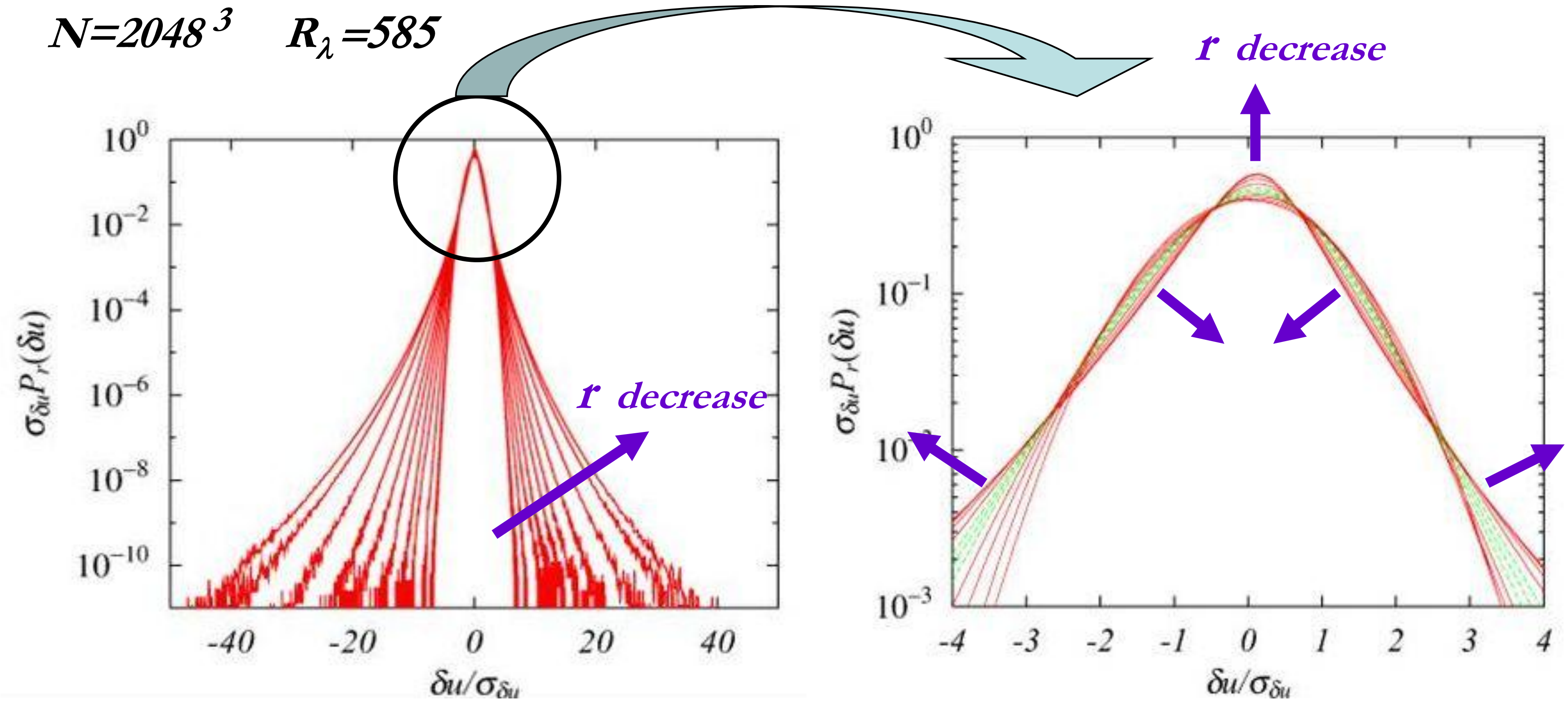


Kolmogorov (K41) $\left\langle U^2\right\rangle \propto r^{2/3}, \quad 2/3 \doteq 0.6667$

DNS (Gotoh et al. 2002)

PDF of longitudinal velocity increment

$N = 2048^3 \quad R_\lambda = 585$



In Kolmogorov Theory

$$(\bar{\epsilon} r)^{2/3} P(U, r) = \bar{P}(U/(\bar{\epsilon} r)^{1/3}) \qquad \eta \ll r \ll L$$

Intermittency

Statistical law changes with scale !

- What is the law of change of PDF with scale?
- How are PDFs predicted from Navier Stokes eq.
- Does the universality exist? If so, where?

What is the statistics of passive scalar?

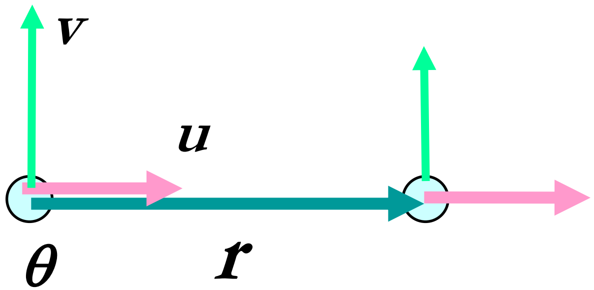
Universality of passive scalar fluctuations at small scales in homogeneous turbulence

Nagoya Inst. Tech.

T. Gotoh, T. Watanabe and Y. Suzuki

Acknowledegment HPC, JHPCN, JSPS, KITP, NIFS, NUCC

Kolmogorov theory for turbulence
Obukhov-Corrsin theory for passive scalar



Velocity and scalar increments

$$U = u(x+r) - u(x), \qquad V = v(x+r) - v(x)$$
$$\Theta = \theta(x+r) - \theta(x)$$

Inertial Range $\eta \ll r \ll L$ Sc=1

PDF

$$(\bar{\epsilon} r)^{2/3} P(U, V, r) = \bar{P}(U/(\bar{\epsilon} r)^{1/3}, V/(\bar{\epsilon} r)^{1/3})$$
$$\bar{\chi}^{1/2} \bar{\epsilon}^{1/6} r^{2/3} Q(U, \Theta, r) = \bar{Q}\left(U/(\bar{\epsilon} r)^{1/3}, \Theta/(\bar{\chi}^{1/2} \bar{\epsilon}^{-1/6} r^{1/3})\right)$$

Moment

$$\langle U^p V^q \rangle \propto r^{\zeta_{p,q}}, \quad \left\langle U^3 \right\rangle = -\frac{4}{5}\bar{\epsilon} r, \qquad \zeta_{p,q} = \frac{p+q}{3}$$
$$\langle \Theta^p U^q \rangle \propto r^{\xi_{p,q}}, \quad \left\langle U\Theta^2 \right\rangle = -\frac{4}{3}\bar{\chi} r, \qquad \xi_{p,q} = \frac{p+q}{3}$$

Scaling exponents are universal ….

Celani et al. (2000, 2001) ; Celani & Vergassola, (2001)
Arad et al. (2001) ; Biferale et al. (2004)…

Is scalar scaling exponent universal ?

Object

to examine the scaling exponents ζ_q^θ for different scalar injection

DNS of passive scalar turbulence

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + \nu \nabla^2 u + f, \quad \nabla \cdot u = 0, \quad \rho = 1$$

$$\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta = \kappa \nabla^2 \theta + f_\theta$$

Velocity: Homogeneous Isotropic Steady

External force: isotropic, Gaussian, white in time, low k band $1 \le |\mathbf{k}| \le 2$

Scalar : Homogeneous Steady

scalar injction:

Case R : isotropic, Gaussian, white in time, low k band $1 \le |\mathbf{k}| \le 2$

Case G : uniform mean scalar gradient $f_\theta = -\Gamma u_3$

Wide range, Anisotropic, Intermittent

DNS parameters

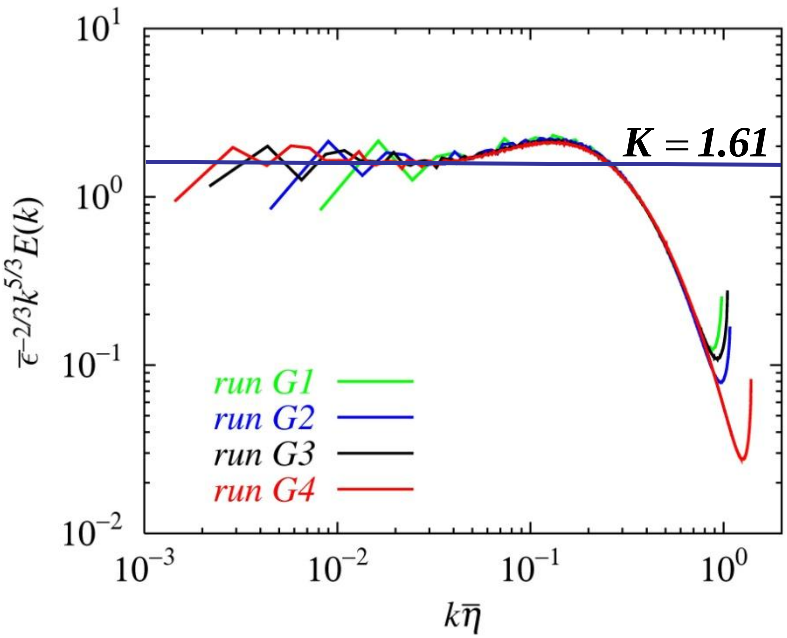
Case R Γ**=0** $S_c = \nu/\kappa = 1$

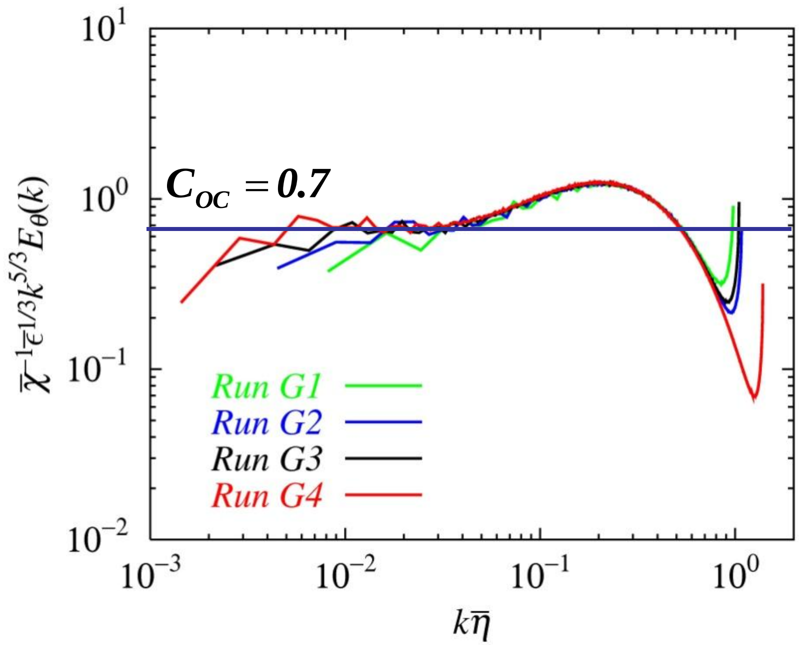
Case G Γ**=1** $S_c = \nu/\kappa = 1$

Run	G1	G2	G3	G4	R1
N^3	256^3	512^3	1024^3	2048^3	2048^3
R_λ	174	263	468	586	688
$\nu(\times 10^{-3})$	1.3	0.60	0.24	0.13	0.13
$K_{max}\bar{\eta}$	0.99	1.09	1.05	1.39	1.36
u'/LT_{av}	27.1	5.62	3.97	2.29	2.75

Spectrum (isotropic sector)

Case G





Isotropic sector

$$E_\theta(k) = C_{OC}\overline{\chi}\overline{\varepsilon}^{-1/3}k^{-5/3}$$

Obukhov-Corrsin constant

$C_{OC} = 0.42 \times 5/3 = 0.7$ **(Case G W&G2007)**

$C_{OC} = 0.68 \pm 0.04$ **(Case R W&G 2004)**

Present $C_{OC}^{1D} = 0.42$

Experiment $C_{OC}^{1D} = 0.45 \sim 0.55$ **Mydlarski & Warhart (1998)**

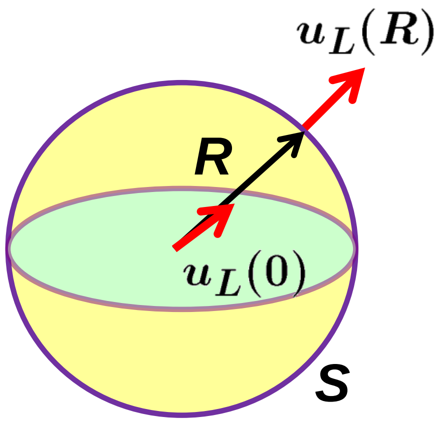
$C_{OC}^{1D} = 0.4$ **Sreenivasan (1996)**

DNS $C_{OC} = 0.87$ **Wang et al. (1999)** $C_{OC}^{1D} = 0.4$ **Yeung et al. (2002)**

For *R* in the inertial convective or viscous convective range

$$\int_{r\le R}\frac{\partial}{\partial r_j}\left\langle \delta u_j(r,t)(\delta\theta(r,t))^2\right\rangle dr = 4\pi R^2\frac{1}{4\pi R^2}\int_{r=R}\left\langle \delta u_j(r,t)(\delta\theta(r,t))^2\right\rangle\frac{r_j}{r}dS$$
$$= 4\pi R^2\left\langle \delta u_L(R,t)(\delta\theta(R,t))^2\right\rangle_{sp}$$
$$= -4\bar{\chi}\frac{4\pi R^3}{3}$$

Spherical average $\langle A\rangle_{\mathrm{sp}} \equiv \frac{1}{4\pi R^2}\int_{r=R} A(r)dS$



$$\left\langle \delta u_L(R,t)(\delta\theta(R,t))^2\right\rangle_{sp} = -\frac{4}{3}\bar{\chi}R$$

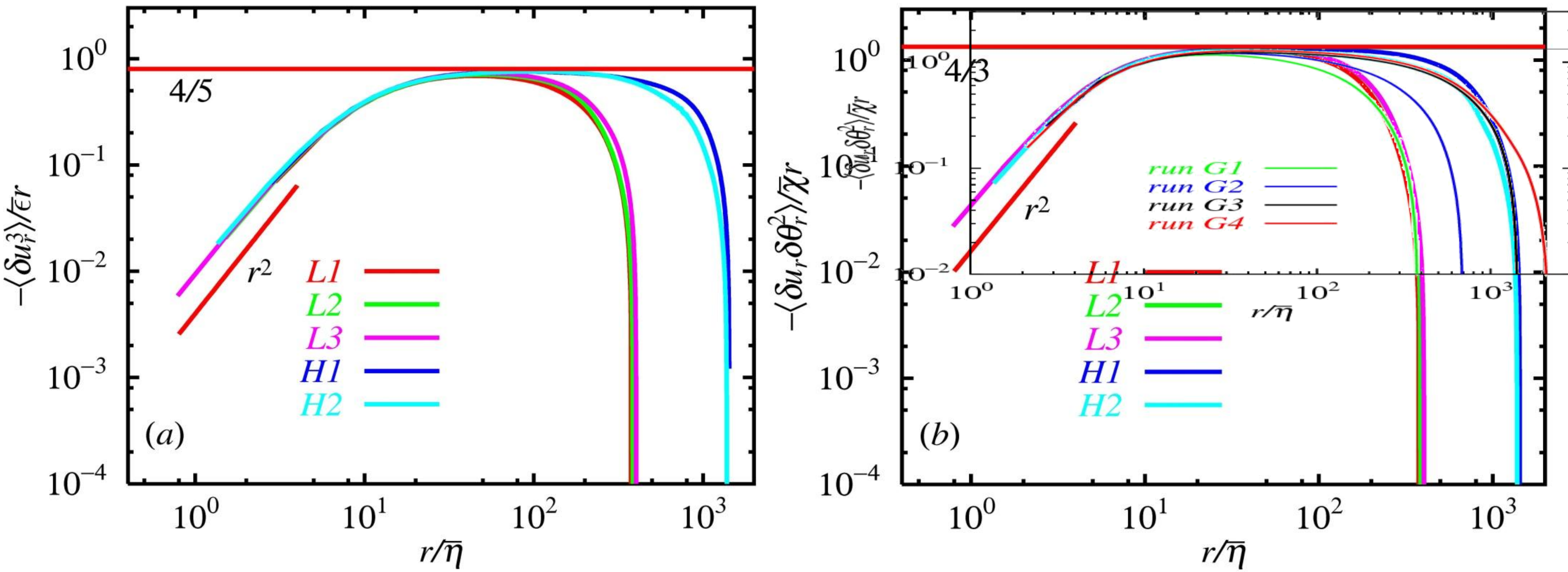
4/3 law holds for the spherical average

Gotoh et al (JOT 2011)

4/5 and 4/3 laws

Case R (isotropic random source)

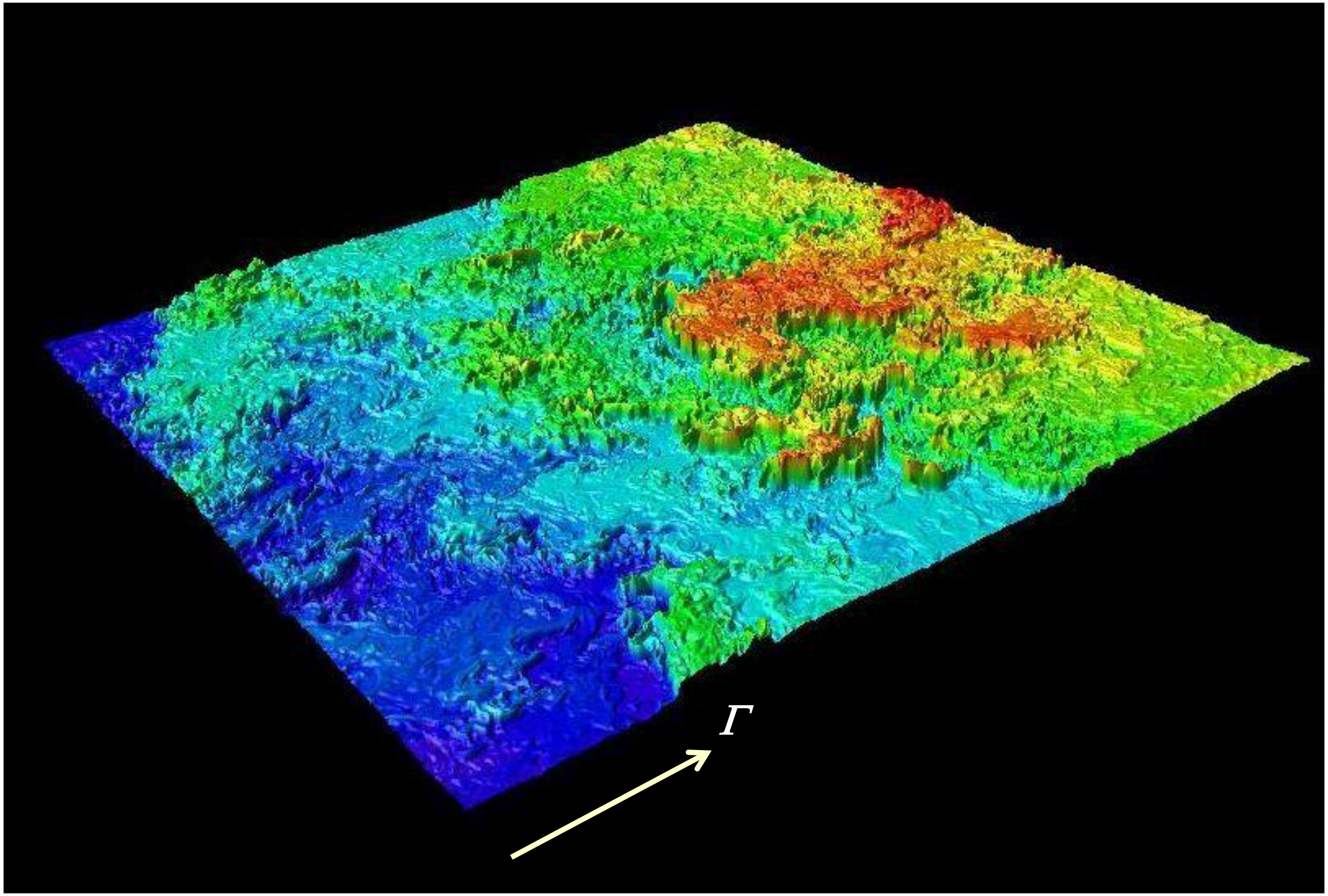
$$-\frac{\langle (\delta u_r)^3 \rangle}{\bar{\epsilon} r} = \frac{4}{5}$$



$$\delta u_r = u(\boldsymbol{x} + r\boldsymbol{e}_x) - u(\boldsymbol{x}) \qquad \delta\theta_r = \theta(\boldsymbol{x} + r\boldsymbol{e}_x) - \theta(\boldsymbol{x})$$

$$\Pi_\theta(k) = -\frac{1}{8\pi^2} \int \frac{\sin(kr)}{r} \frac{\partial}{\partial r_l} \left(\frac{r_l}{r^2} \frac{\partial}{\partial r_j} \langle \delta u_j(r) (\delta\theta(r))^2 \rangle \right) \, \mathrm{d}\boldsymbol{r} = \bar{\chi}$$

$T = \langle T \rangle + \theta = \Gamma z + \theta$

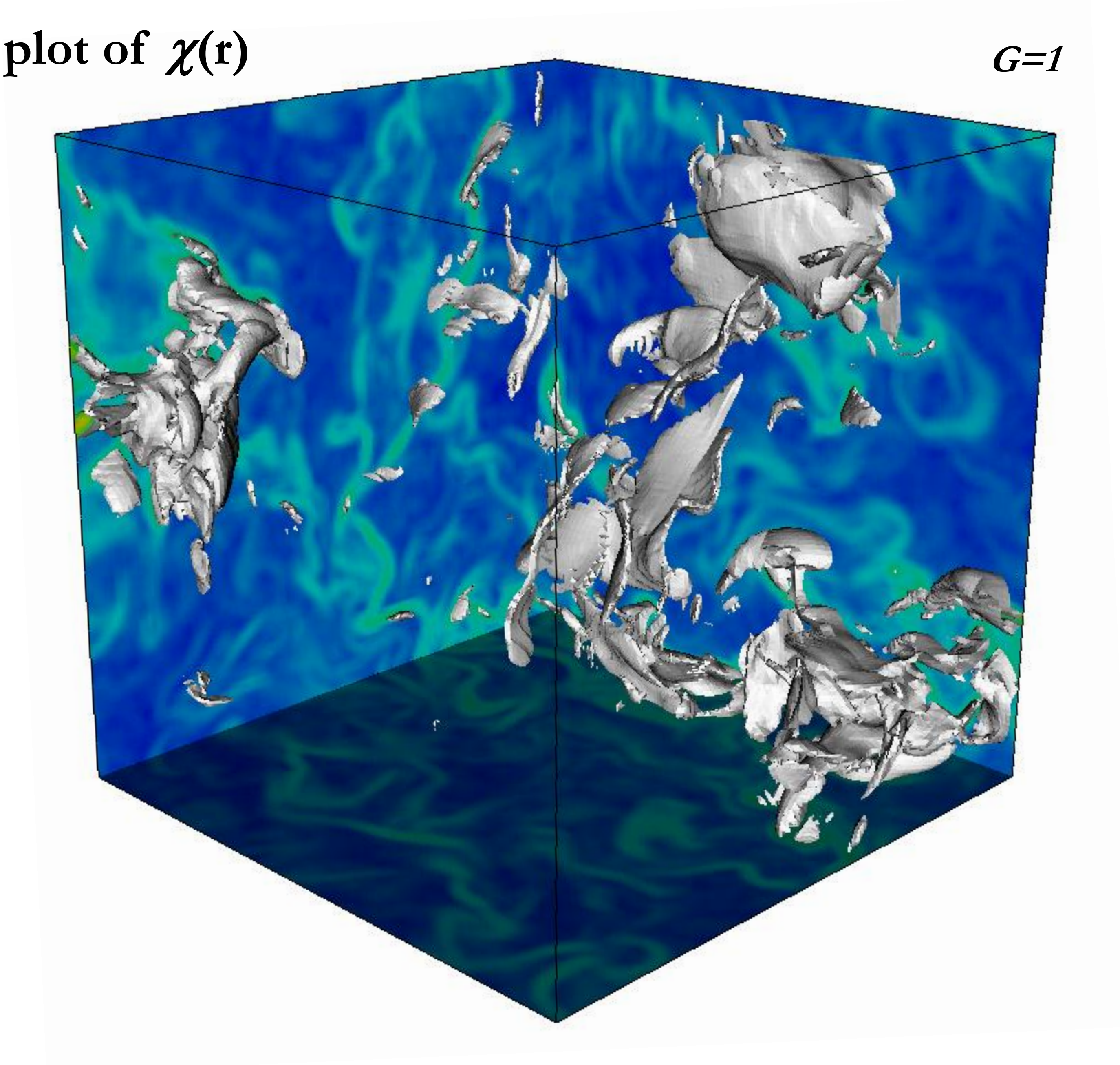


$R_{\lambda} = 468$, $Sc=1$, $N=1024^{3}$

Gotoh and Watanabe (Physica D 2010)

3D plot of $\chi(r)$

G=1



Scaling exponents of structure functions

In the inertial convective range $\eta << r << L$

$$S_q^L(r) = \langle (\delta u(r))^q \rangle \propto r^{\zeta_q^L}, \qquad \delta u(r) = (u(x+r) - u(x)) \cdot r/r$$

$$S_q^T(r) = \langle (\delta v(r))^q \rangle \propto r^{\zeta_q^T}, \qquad \delta u(r) = (u(x+r) - u(x)) \cdot (I - rr/r)$$

$$S_q^\theta(r) = \langle (\delta \theta(r))^q \rangle \propto r^{\zeta_q^\theta}, \qquad \delta \theta(r) = \theta(x+r) - \theta(x)$$

$$S_q^{\theta L}(r) = \left\langle (\delta u(r) \delta \theta(r)^2)^q \right\rangle \propto r^{\zeta_q^{\theta L}}$$

Local scaling exponent

$$\zeta_q^\alpha(r) = \frac{\mathrm{d} \ln S_q^\alpha(r)}{\mathrm{d} \ln r}, \qquad \alpha = L,\ T,\ \theta,\ \theta L$$

Anisotropy

Case G

Scalar injection statistics

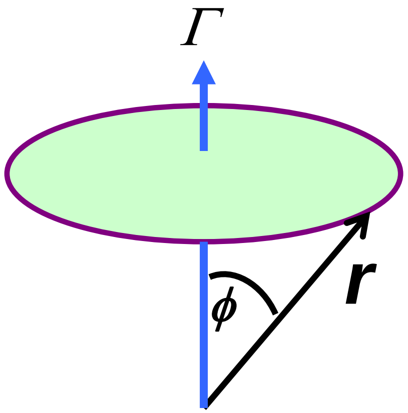
$$f_\theta(x,t) = -\Gamma u_3(x,t)$$

axisymmetric reflection invariant wide range intermittent

$$\langle f_\theta(x,t)\rangle = 0$$

$$\langle f_\theta(x+r,t) f_\theta(x,t)\rangle = \Gamma^2 \langle u_3(x+r,t) u_3(x,t)\rangle$$

$$= \Gamma^2 u'^2 \left(\frac{1}{3r^2}\frac{d}{dr}(r^3 f) - \frac{r}{3}\frac{df(r)}{dr} P_2(\cos\phi) \right)$$



Scalar statistics

axisymmetric reflection invariant

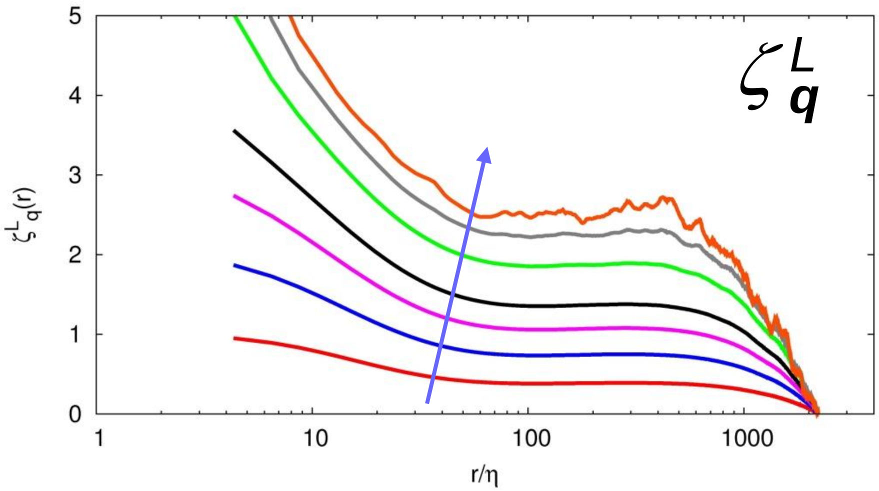
$$S_{2q}(r,\phi) = \sum_{l=0}^{\infty} S_{2q}^{(2l)}(r) P_{2l}(\cos\phi)$$

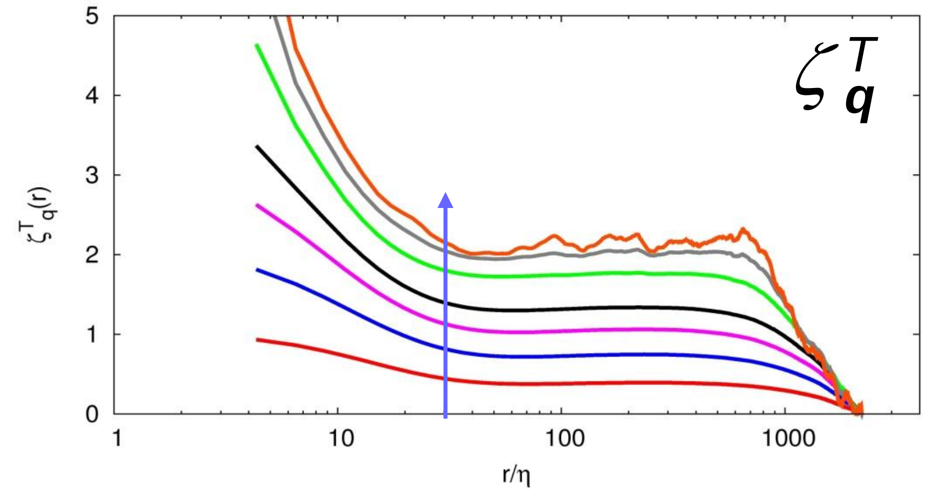
$$= \underline{S_{2q}^{(0)}(r)} + S_{2q}^{(2)}(r) P_2(\cos\phi) + S_{2q}^{(4)}(r) P_4(\cos\phi) \cdots$$

Isotropic sector

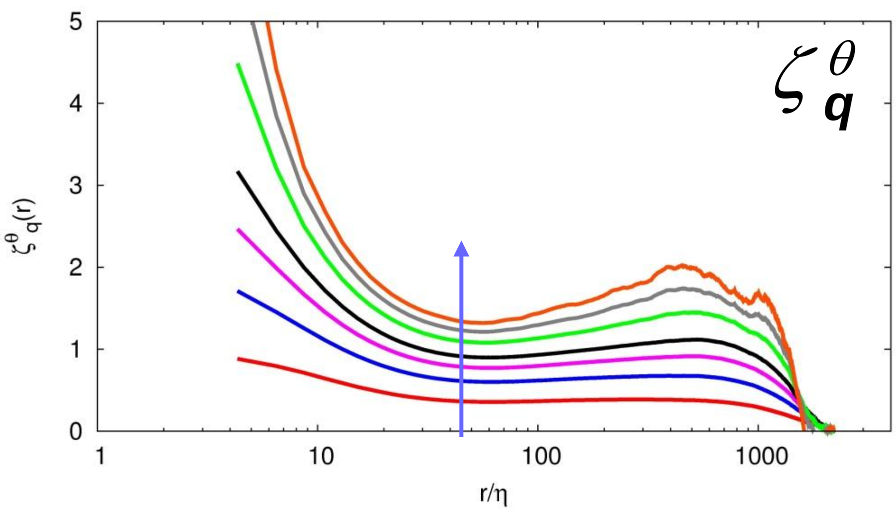
Local scaling exponents

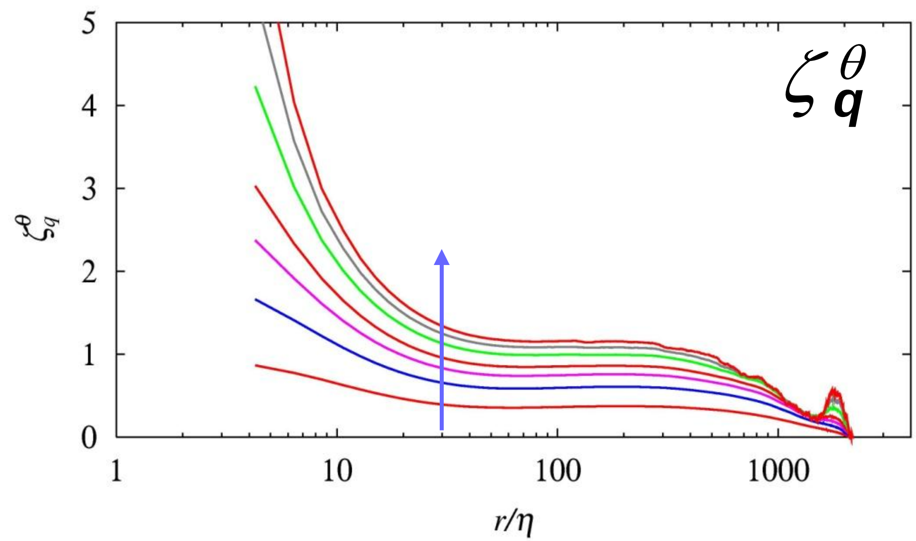
q=1,2,3,4,6,8,10 $R_\lambda = 688$ *Sc=1*





Case R **Case G**





Observation

Case G curves are lowest and tend to saturate

Crossover length

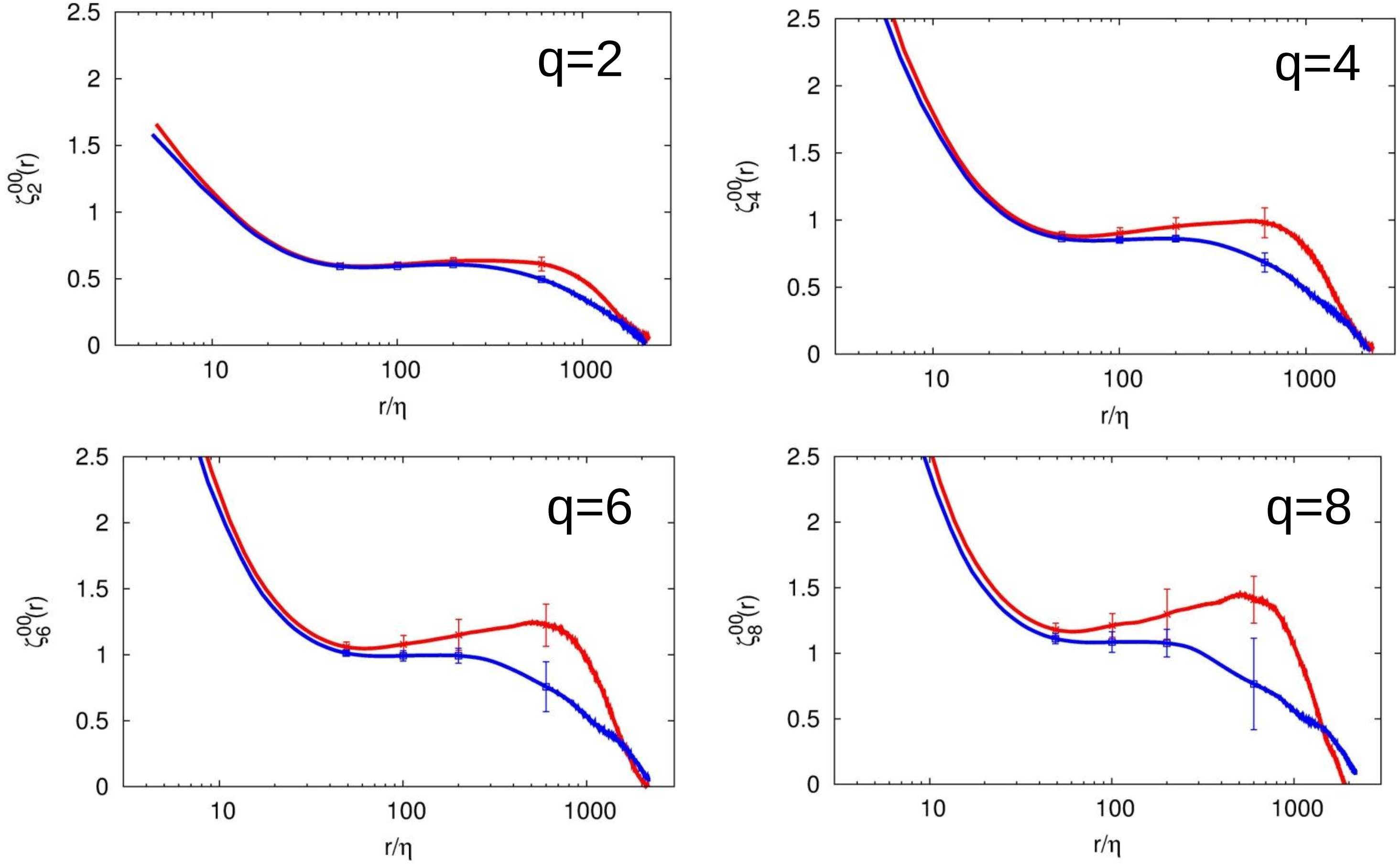
L: increase with q

T, q: unchange

Local scaling exponents

$R_\lambda = 688 \quad Sc=1$

Case R — Case G (isotropic sector)



Local scaling exponents in the inertial –convective range are not equal at high order !

Gotoh et al (JOT 2011)

Summary

Small scale fluctuations

- Anisotropy in the scalar field in case G (uniform gradient) is weak and mostly from *l=2* sector

- Intermittency in case G is stronger than in case R

Universality of scalar scaling exponents

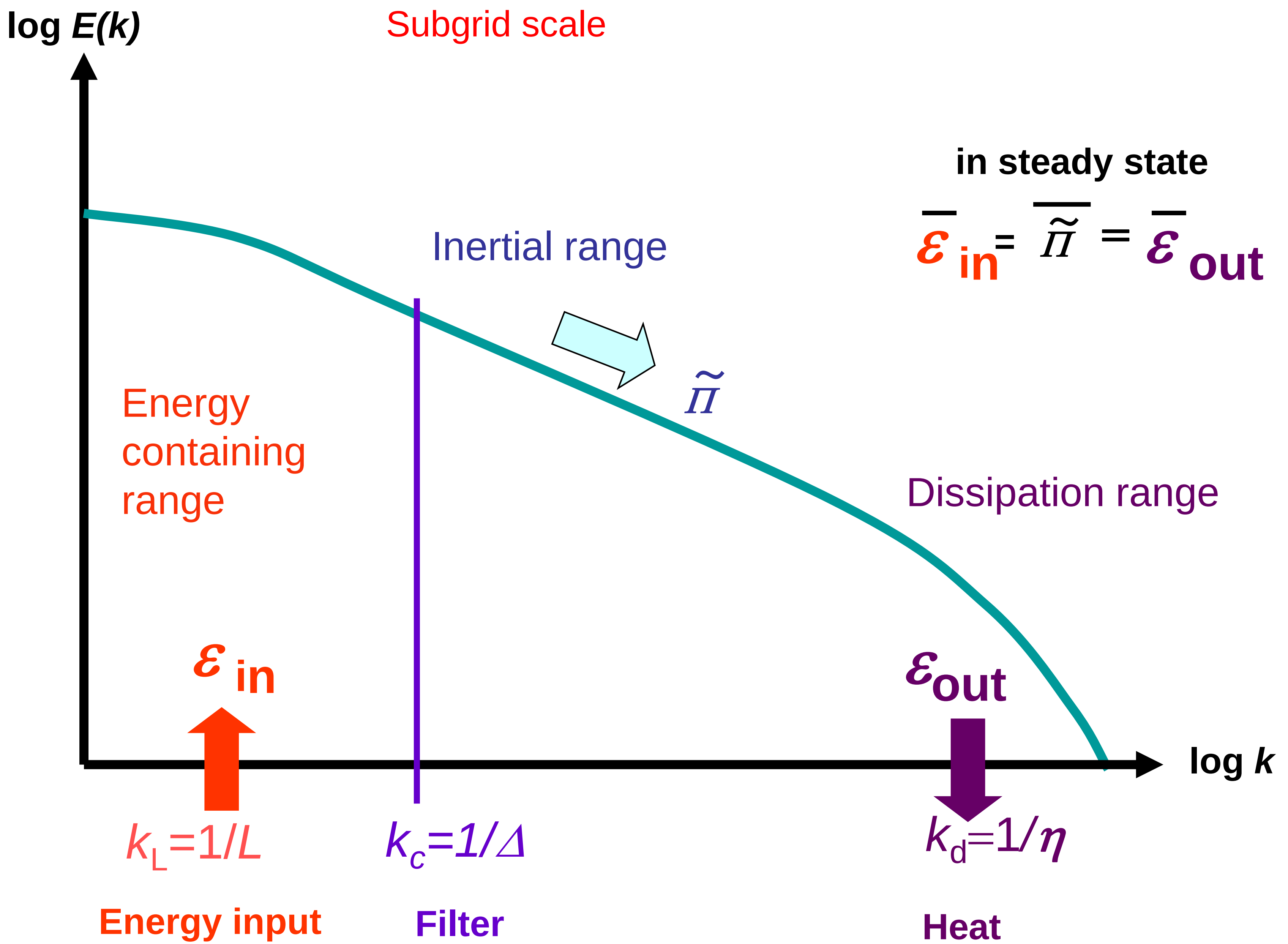
1: There is no universality of the scaling exponents of high order structure function

2: The Reynolds number is too low to observe the asymptotic scaling exponents

Eddy damping, Vertex correction, and Langevin modeling for homogeneous isotropic turbulence

T. Gotoh (Nagoya Inst. of Tech.)
R. Rubinstein (NASA Langley)
W. Bos (CNRS,Lyon)
S. Hatanaka (Nagoya Inst. of Tech.)

Acknowledgement NIFS, NUCC


log E(k)
Subgrid scale
in steady state
ε̄in = Π̃̄ = ε̄out
Inertial range
Π̃
Energy containing range
Dissipation range
ε in
εout
log k
kL=1/L
kc=1/Δ
kd=1/η
Energy input
Filter
Heat

Navier-Stokes Eq.

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) u(k,t) = M(k) : \sum_{k=p+q} u(p,t)u(q,t) \equiv N(k,t)$$

Filtering

$$\begin{aligned} u(k,t) &= \mathcal{P}(k)u(k,t) + (1-\mathcal{P}(k))u(k,t) \qquad \mathcal{P}(k) = H(k_c - |k|) \\ &= u^<(k,t) + u^>(k,t) \end{aligned}$$

Equation of GS field

$$\left(\frac{\partial}{\partial t} + \nu k^2\right) u^<(k,t) = N^<(k,t) + S^<(k,t)$$

$$N^<(k,t) = \mathcal{P}(k) \sum M(k) : u^<(p,t)u^<(q,t),$$

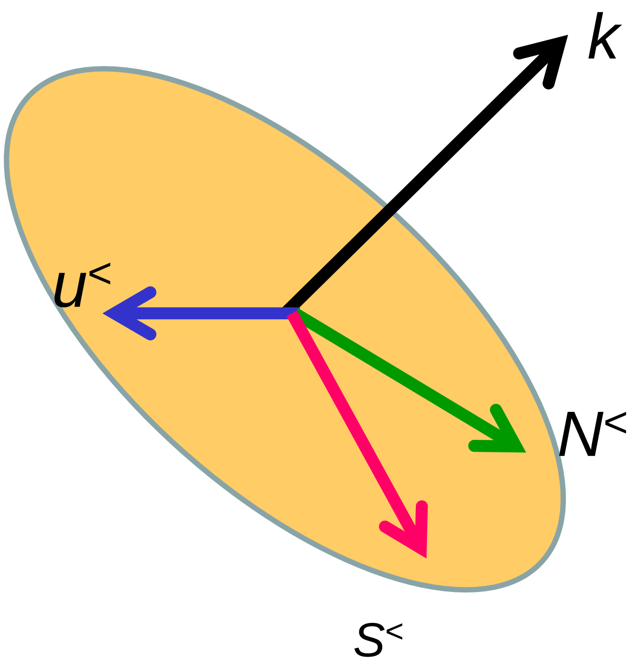
$$S^<(k,t) = \mathcal{P}(k)N(k,t) - N^<(k,t)$$

SGS contributions

Statistical projection

$$S^{<}(k,t) = \underline{C_1(k,t)u^{<}(k,t)} + \underline{C_2(k,t)N^{<}(k,t)} + \underline{R^{<}(k,t)}$$

Correlated with $u^{<}$ and $N^{<}$ Uncorrelated part



Assumption

C_1 and C_2 are scalar functions of $|k|$

Kraichnan(1976)
Domaradzki et al. (1987)
Chasnov (1991)
Metais & Lesieur (1992)

Langford & Moser (1999,2004)

Statistical projection

Eddy damping Vertex correction

$$S^{<}(k,t) = \underline{C_1(k,t)u^{<}(k,t)} + \underline{C_2(k,t)N^{<}(k,t)} + \underline{R^{<}(k,t)}$$

Correlated with $u^<$ and $N^<$ Uncorrelated part

$$\begin{pmatrix} \langle u^{<}(k,t)\cdot u^{<}(-k,t)\rangle & \langle N^{<}(k,t)\cdot u^{<}(-k,t)\rangle \\ \langle u^{<}(k,t)\cdot N^{<}(-k,t)\rangle & \langle N^{<}(k,t)\cdot N^{<}(-k,t)\rangle \end{pmatrix} \begin{pmatrix} C_1(k,t) \\ C_2(k,t) \end{pmatrix} = \begin{pmatrix} \langle S^{<}(k,t)\cdot u^{<}(-k,t)\rangle \\ \langle S^{<}(k,t)\cdot N^{<}(-k,t)\rangle \end{pmatrix}$$

$$\begin{pmatrix} C_1(k,t) \\ C_2(k,t) \end{pmatrix} = J^{-1}(k,t) \begin{pmatrix} K(k,t) & -H(k,t) \\ -H(k,t) & Q(k,t) \end{pmatrix} \begin{pmatrix} D(k,t) \\ F(k,t) \end{pmatrix}$$

where

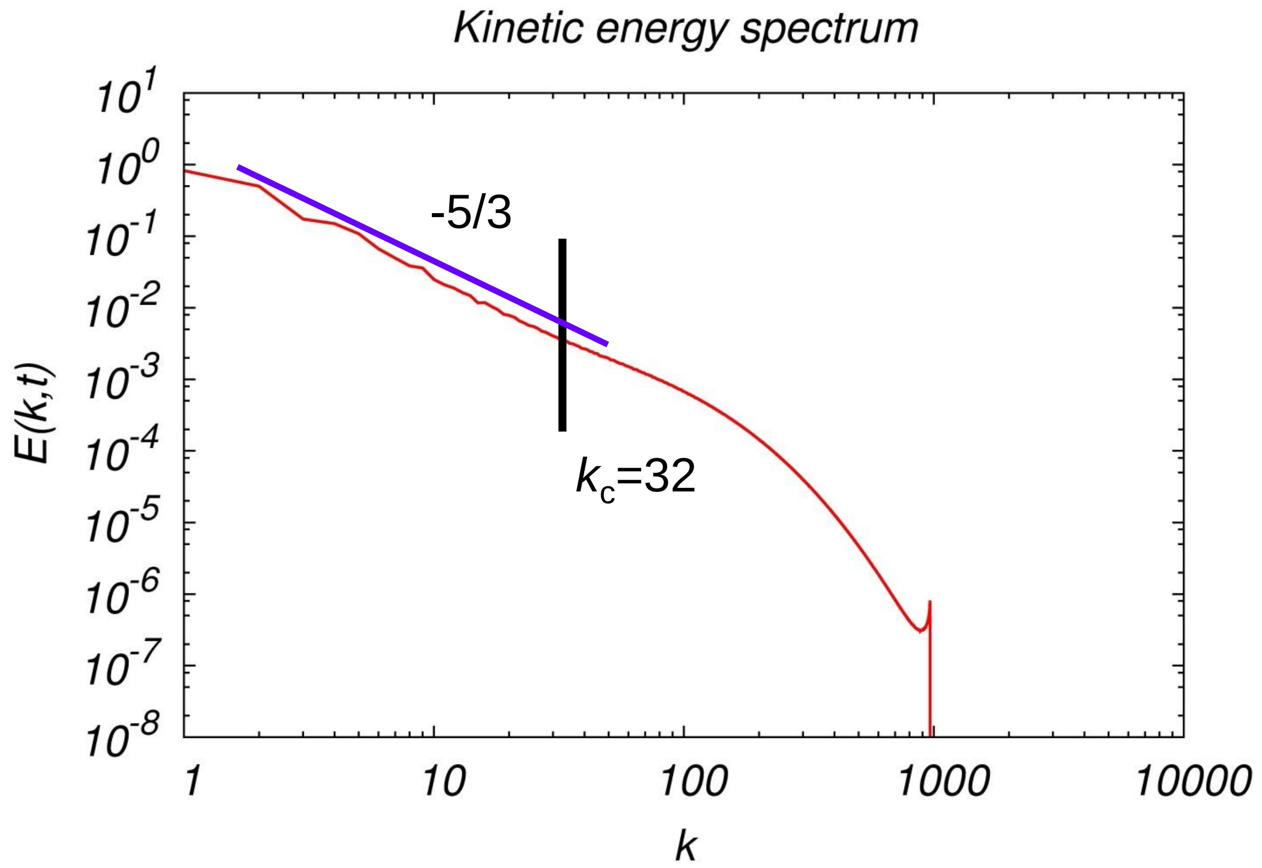
$$J(k,t) = Q(k,t)K(k,t) - H(k,t)^2, \qquad Q(k,t) = \langle u^{<}(k,t)\cdot u^{<}(-k,t)\rangle$$

$$H(k,t) = \langle N^{<}(k,t)\cdot u^{<}(-k,t)\rangle, \qquad K(k,t) = \langle N^{<}(k,t)\cdot N^{<}(-k,t)\rangle$$

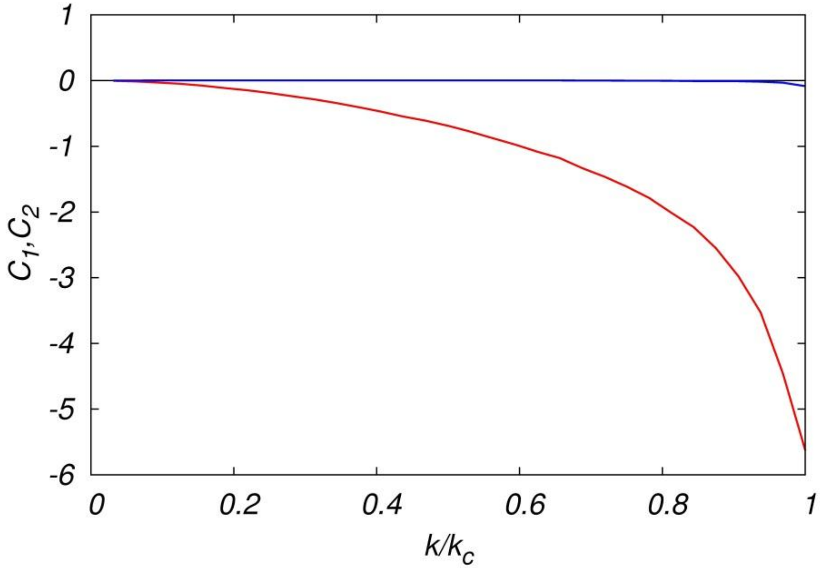
$$D(k,t) = \langle S^{<}(k,t)\cdot u^{<}(-k,t)\rangle, \qquad F(k,t) = \langle S^{<}(k,t)\cdot N^{<}(-k,t)\rangle$$

DNS of steady turbulence

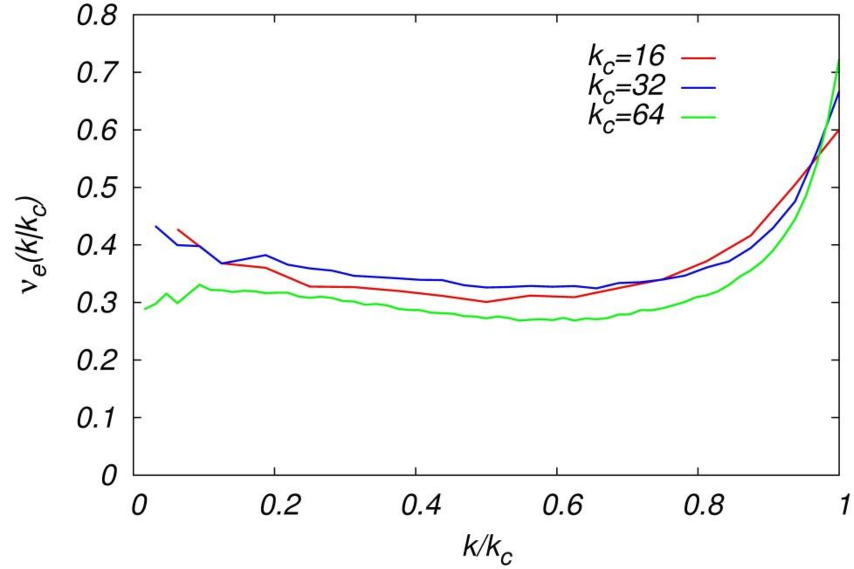
$$N = 2048^3,\ R_\lambda \approx 690,\ k_0 = 1,\ T_{av} = 1.8T_{eddy},\ k_c = 16, 32, 64$$

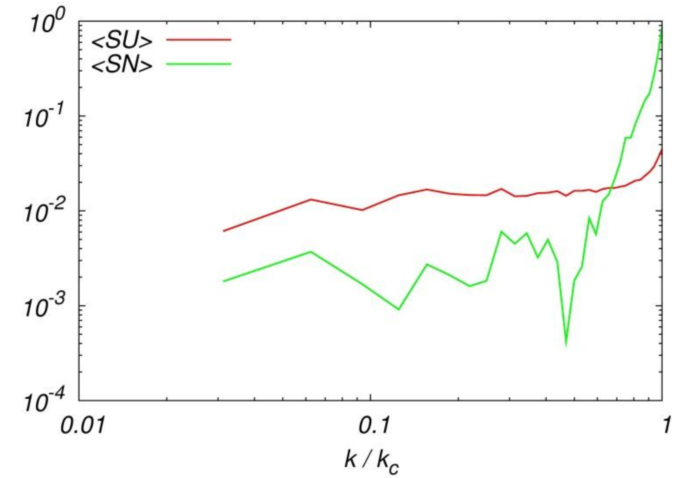


C_1 and C_2



$\nu_e(k|k_c)$





$\nu_e(k|k_c)$ tends to constant as $k/k_c \longrightarrow 0$

C_2 term is very small (negligible vertex correction)
(but does not mean that $\langle S^< N^< \rangle$ is small)

What is the statistics of the uncorrelated part?

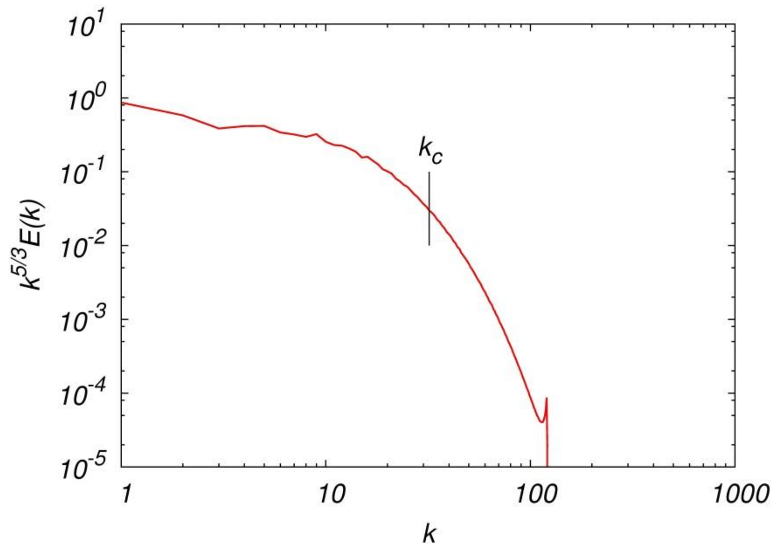
$$\langle R^{<}(k,t)\rangle = 0$$

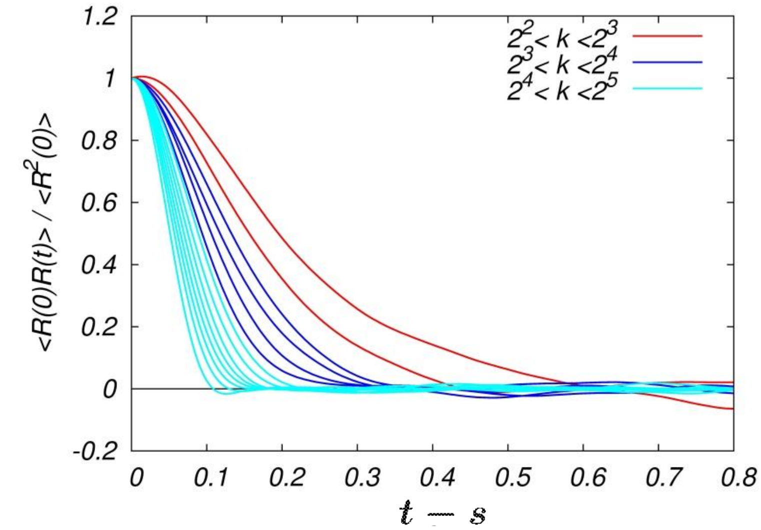
$$\langle R^{<}(k,t)R^{<}(-k,s)\rangle = F(k,(t-s)/\tau_k) \qquad \tau_k = ?$$

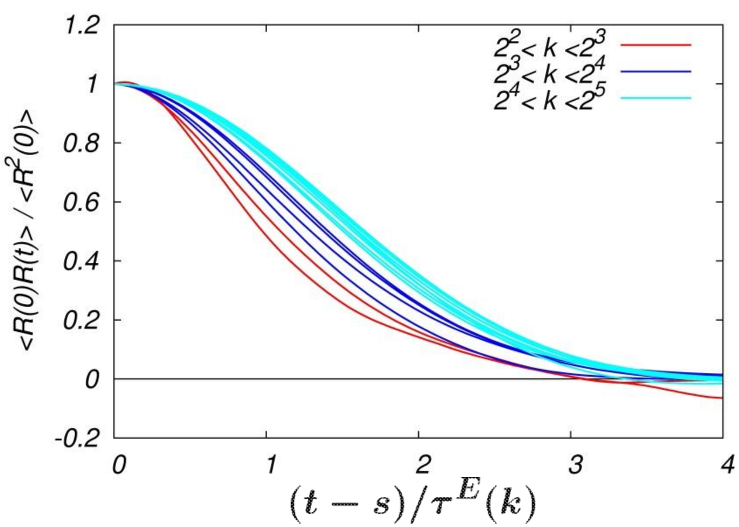
PDF of $R^{<}(k,t)$?

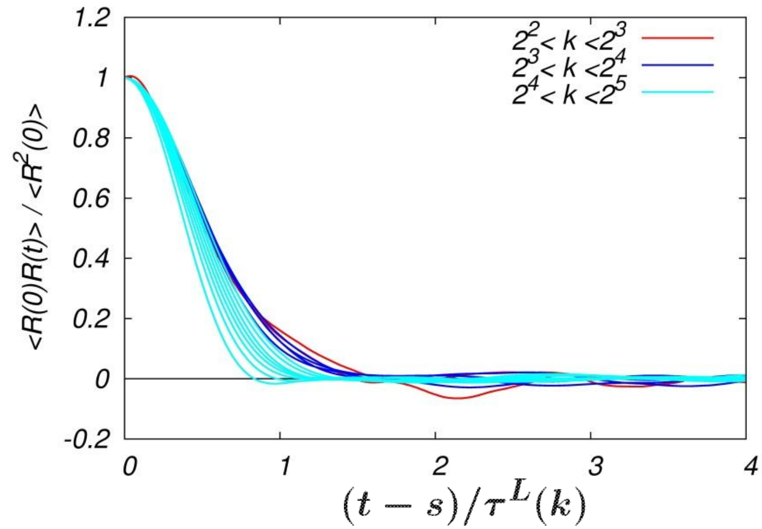
Two time correlation in steady turbulence

$$N = 256^3, \; R_\lambda \approx 120, \quad k_c = 32$$









$$\tau^E(k) = (ku_0)^{-1}$$

$$\tau^L(k) = \left(\int_0^k p^2 E(p) dp \right)^{-1/2}$$

Langevin modeling of SGS contributions

Eddy damping + random force

$$\begin{aligned} S^{<}(k,t) &= C_1(k,t)u^{<}(k,t) + C_2(k,t)N^{<}(k,t) + R^{<}(k,t) \\ &\approx C_1(k,t)u^{<}(k,t) + R^{<}(k,t) \end{aligned}$$

Decorrelation time of the random force is given by Lagrangian time
(sweeping effects are not dominant)

Future problem

Is PDF of the random part Gaussian ?

What is the statistical correlation at two time ?

$$S^{<}(k,t) = \int_{-\infty}^{t} C_1(k,t,s)u^{<}(k,s)ds + \int_{-\infty}^{t} C_2(k,t,s)N^{<}(k,s)ds + R^{<}(k,t)$$

Acknowledgments

NIFS and NUCC