Nonlinear Gyrokinetic Description of Alfvénic Turbulence

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Outline

- Introduction
- Gyrokinetics and Polarization Density
- Alfvén Eigenmodes in Tokamaks
- Weak Turbulence Theory of Toroidal Alfvén Eigenmodes
- Relations to MHD Turbulence

Basics of Shear- Alfvén Waves

$$\delta E_{\parallel} = 0 \qquad \Rightarrow \frac{1}{c}\partial_t \delta A_{\parallel} = -\nabla_{\parallel}\delta\phi$$

ideal MHD

From $\vec{\nabla}\cdot\vec{\delta} J = 0$

$$\Rightarrow \frac{m_i n_0 c}{B^2}\partial_t \nabla_{\perp}{}^2 \delta\phi = \nabla_{\parallel} J_{\parallel} = \nabla_{\parallel} \nabla_{\perp}{}^2 \delta A_{\parallel}$$

$$\mathrm{v_A}^{-2}\omega^2 = k_{\parallel}^2$$

MHD : from $\vec{\nabla}\times$ of $\rho\frac{d}{dt}\vec{\mathrm{v}} = \vec{J}\times\vec{B}$

i.e. " Vorticity "

Gyrokinetics : " Polarization Density "

$$n_{\mathrm{pol}} = \nabla_{\perp}\bullet\left(\frac{m_i n_0 c}{B^2}\nabla_{\perp}\delta\phi\right)$$

Gyrokinetics and Polarization Density

Ref. Hahm-Lee-Brizard, Phys Fluids ‘88

- Vorticity Equation can be derived from

$$\frac{dn_e}{dt} = \frac{1}{|e|}\nabla_\parallel j_{\parallel e}$$

$$\frac{d}{dt}N_{\text{gyrocenter}} = -\frac{1}{|e|}\nabla_\parallel j_{\parallel i}$$

⬅ From moments of Gyrokinetic Equation

$-$

$$\frac{d}{dt}\,\text{"}N_{\text{pol}}\text{"} = \frac{1}{|e|}\nabla_\parallel J_\parallel$$

- What is Physical Meaning of Polarization Density?
- What is Gyrokinetics?

Conventional (old-fashioned) Derivation of Non-linear Gyrokinetic Equation

- Closely follow Guiding Center transformation by
  P.J. Catto, Plasma Phys. **20**, 719 (1977)
- Resulting equation
  Frieman and Chen, PF **25**, 502 (1982)
  Lee, PF **26** 556 (1983)
- Purpose: illustrate basic physics and
  mathematical complexity
  involved in this conventional method.

Consider uniform $\mathbf{B} = B\hat{\mathbf{b}}$ to emphasize nonlinear effects

- Goal: from

$$\left[\frac{\partial}{\partial t} + \mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} + \frac{q}{m}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}\right)\cdot\frac{\partial}{\partial \mathbf{v}}\right] f(\mathbf{x},\mathbf{v},t) = 0 \quad \text{6D Vlasov Eqn}$$

get

$$\left(\frac{\partial}{\partial t} + \frac{d\mathbf{R}}{dt}\cdot\frac{\partial}{\partial \mathbf{R}} + \frac{dv_\parallel}{dt}\frac{\partial}{\partial v_\parallel}\right)\langle f\rangle(\mathbf{R},\mu,v_\parallel,t) = 0 \quad \text{5D GK Eqn}$$

with

$$\frac{d\mu}{dt} = 0 \text{ and } \frac{\partial}{\partial\theta}\langle f\rangle = 0$$

$\mu \simeq v_\perp^2/(2B)$: magnetic moment, an adiabatic invariant at lowest order

- Assumption:
  - $\omega \ll \Omega_{ci}$
  - $k_\parallel \ll k_\perp \sim \rho_i^{-1}$
  - $\delta f/f_0 \sim \delta n/n_0 \sim e\delta\phi/T_e \ll 1$

Guiding Center Transformation à la Catto

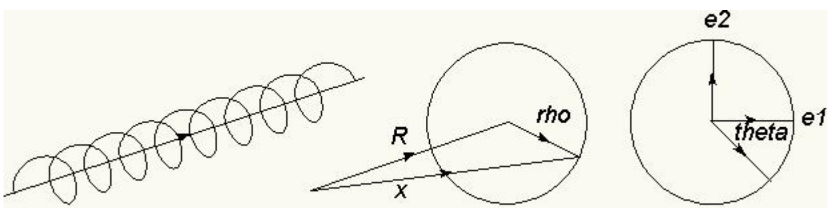
$(\mathbf{x}, \mathbf{v}) \to (\mathbf{R}, \mathbf{v}_\parallel, \mu, \theta)$, θ : gyrophase-angle

$\mathbf{R} = \mathbf{x} - \boldsymbol{\rho}$, $\boldsymbol{\rho} = \frac{\hat{\mathbf{b}} \times \mathbf{v}}{\Omega}$, $\Omega = \frac{eB}{mc}$

$v_\parallel = \hat{\mathbf{b}} \cdot \mathbf{v}$, $\mu = v_\perp^2/(2B)$

θ defined by

$$\begin{cases} \mathbf{v} = v_\parallel \hat{\mathbf{b}} + v_\perp \hat{\mathbf{e}}_\perp \\ \hat{e}_\perp = -\hat{\mathbf{e}}_2 \cos\theta - \hat{\mathbf{e}}_1 \sin\theta \\ \hat{\mathbf{e}}_\rho = \hat{\mathbf{e}}_1 \cos\theta - \hat{\mathbf{e}}_2 \sin\theta \end{cases}$$



Note that for uniform $\mathbf{B}$,

$$d^3\mathbf{x} d^3\mathbf{v} = \underbrace{B} \, d\mu d\theta dv_\parallel d^3\mathbf{R}$$

B : “phase-space volume”

Then, we would like to express $\frac{\partial}{\partial \mathbf{x}}$ and $\frac{\partial}{\partial \mathbf{v}}$ in G.C. space i.e., in terms of $\mu, v_{\|}, \mathbf{R},$ and θ;

$$\frac{\partial}{\partial \mathbf{x}} = \frac{\partial \mathbf{R}}{\partial \mathbf{x}} \cdot \frac{\partial}{\partial \mathbf{R}} + \frac{\partial \mu}{\partial \mathbf{x}} \frac{\partial}{\partial \mu} + \frac{\partial v_{\|}}{\partial \mathbf{x}} \frac{\partial}{\partial v_{\|}} + \frac{\partial \theta}{\partial \mathbf{x}} \cdot \frac{\partial}{\partial \theta}$$

$$\frac{\partial}{\partial \mathbf{v}} = \frac{\partial \mathbf{R}}{\partial \mathbf{v}} \cdot \frac{\partial}{\partial \mathbf{R}} + \frac{\partial \mu}{\partial \mathbf{v}} \frac{\partial}{\partial \mu} + \frac{\partial v_{\|}}{\partial \mathbf{v}} \frac{\partial}{\partial v_{\|}} + \frac{\partial \theta}{\partial \mathbf{v}} \cdot \frac{\partial}{\partial \theta}$$

$\rightarrow$ important to check what quantities are held constant when taking partial derivatives

Since

$$\frac{\partial}{\partial \mathbf{x}} \mu \bigg|_{\mathrm{v=const}} = 0, \; \frac{\partial}{\partial \mathbf{x}} v_{\|} \bigg|_{\mathrm{v=const}} = 0, \; \frac{\partial}{\partial \mathbf{x}} \bigg|_{\mathrm{v=const}} \theta = 0, \text{ and } \mathbf{R} = \mathbf{x} - \frac{\hat{\mathbf{b}} \times \mathbf{v}}{\Omega}$$

$\frac{\partial}{\partial \mathbf{x}} \rightarrow$ only the 1st term on the R.H.S. survives $\Rightarrow$

$$\frac{\partial}{\partial \mathbf{x}} = \mathbf{I} \cdot \frac{\partial}{\partial \mathbf{R}} = \frac{\partial}{\partial \mathbf{R}}$$

Also, noting that

$$\left.\frac{\partial}{\partial \mathbf{v}}\right|_{\mathrm{x=const}} v_{\|} = \left.\frac{\partial}{\partial \mathbf{v}}\right|_{\mathrm{x=const}} \mathbf{v}\cdot\hat{\mathbf{b}} = \hat{\mathbf{b}}, \quad \frac{\partial}{\partial \mathbf{v}}\mu = \mathbf{v}_{\perp}/B$$

$$\frac{\partial}{\partial \mathbf{v}}\mathbf{R} = \frac{\partial}{\partial \mathbf{v}}(\mathbf{x} - \frac{\hat{\mathbf{b}}\times\mathbf{v}}{\Omega}) \rightarrow -\frac{\partial}{\partial \mathbf{v}}(\frac{\hat{\mathbf{b}}\times\mathbf{v}}{\Omega}) = \frac{\mathbf{I}\times\hat{\mathbf{b}}}{\Omega}$$

$$\frac{\partial}{\partial \mathbf{v}} = \hat{\mathbf{b}}\frac{\partial}{\partial v_{\|}} + \frac{\mathbf{v}_{\perp}}{B}\frac{\partial}{\partial \mu} - \frac{\hat{\mathbf{b}}\times\hat{\mathbf{e}}_{\perp}}{v_{\perp}}\frac{\partial}{\partial \theta} + \frac{\mathbf{I}\times\hat{\mathbf{b}}}{\Omega}\frac{\partial}{\partial \mathbf{R}}$$

$\Rightarrow$

$$\mathbf{v}\cdot\frac{\partial}{\partial \mathbf{x}} = v_{\|}\hat{b}\cdot\frac{\partial}{\partial \mathbf{R}} + \mathbf{v}_{\perp}\cdot\frac{\partial}{\partial \mathbf{R}} \tag{1}$$

$$\frac{q}{m}\mathbf{E}\cdot\frac{\partial}{\partial \mathbf{v}} = \frac{q}{m}\left(E_{\|}\cdot\frac{\partial}{\partial v_{\|}} + \frac{\mathbf{E}\cdot\mathbf{v}_{\perp}}{B}\frac{\partial}{\partial \mu} - \frac{\mathbf{E}\cdot\hat{\mathbf{b}}\times\mathbf{v}_{\perp}}{v_{\perp}^2}\frac{\partial}{\partial \theta}\right) + \underline{\frac{c\mathbf{E}\times\mathbf{B}}{B^2}\cdot\frac{\partial}{\partial \mathbf{R}}} \tag{2}$$

$$\frac{q\mathbf{v}\times\mathbf{B}}{mc}\cdot\frac{\partial}{\partial \mathbf{v}} = 0 + 0 - \Omega\frac{\mathbf{v}\times\mathbf{B}\cdot\mathbf{B}\times\mathbf{v}_{\perp}}{B^2 v_{\perp}^2}\frac{\partial}{\partial \theta} + \Omega\frac{(\mathbf{v}\times\hat{\mathbf{b}})\times\hat{\mathbf{b}}}{\Omega}\cdot\frac{\partial}{\partial \mathbf{R}}$$

$$= \Omega\frac{\partial}{\partial \theta} - \underline{\mathbf{v}_{\perp}\cdot\frac{\partial}{\partial \mathbf{R}}} \tag{3}$$

We also want to express $\phi(\mathbf{x})$ and $\mathbf{E}(\mathbf{x})$ in terms of $(\mathbf{R}, \mu, \mathbf{v}_{\|}, \theta)$

$$\phi(\mathbf{x}) = \phi(\mathbf{R} + \boldsymbol{\rho}(\theta)) \Rightarrow$$

$$\frac{\partial \phi}{\partial \theta} = \left.\frac{\partial \mathbf{x}}{\partial \theta}\right|_{\mathbf{R}} \cdot \frac{\partial \phi}{\partial \mathbf{x}} = \frac{\partial \boldsymbol{\rho}}{\partial \theta} \cdot \frac{\partial \phi}{\partial \mathbf{x}} = \frac{\mathbf{v}_\perp}{\Omega} \cdot \frac{\partial \phi}{\partial \mathbf{x}} = -\frac{\mathbf{E} \cdot \mathbf{v}_\perp}{\Omega}$$

∴ the 2nd term of RHS of Eq. (4.2)

$$\frac{q}{m} \frac{\mathbf{E} \cdot \mathbf{v}_\perp}{B} \frac{\partial}{\partial \mu} = -\frac{1}{c} (\frac{q}{m})^2 \frac{\partial \phi}{\partial \theta} \frac{\partial}{\partial \mu}$$

Collecting all terms in Eqs. (1)-(3),

$$\left[\frac{\partial}{\partial t} + v_{\|} \hat{\mathbf{b}} \cdot \frac{\partial}{\partial \mathbf{R}} + c \frac{\mathbf{E} \times \mathbf{B}}{B^2} \cdot \frac{\partial}{\partial \mathbf{R}} - \frac{q}{m} \nabla_{\|} \phi \frac{\partial}{\partial v_{\|}} + \Omega \frac{\partial}{\partial \theta} - \frac{q\Omega}{mB} \frac{\partial \phi}{\partial \theta} \frac{\partial}{\partial \mu} - \Omega \frac{\mathbf{v}_E \cdot \mathbf{v}_\perp}{v_\perp^2} \frac{\partial}{\partial \theta}\right] f = 0 \tag{4}$$

$$-i\omega \qquad ik_{\|} v_{\|} \qquad \mathbf{k}_\perp \cdot \mathbf{v}_E \qquad k_{\|} v_{\|} \left(\frac{e\phi}{T_e}\right) \qquad \Omega \qquad \underbrace{(i) \qquad\qquad (ii)}_{\text{ugly!}}$$

- Term (i) can be shown to be the 1st order correction to μ

i.e.,

$$\frac{d\mu}{dt} = \frac{d\mu^{(0)}}{dt} + \frac{d\mu^{(1)}}{dt} \Rightarrow \frac{d}{dt}(\frac{v_\perp^2}{2B})^{(1)} = \frac{\mathbf{v}_\perp^{(0)}}{B} \cdot \frac{d}{dt}\mathbf{v}_\perp^{(1)}(\theta)$$

where

$$\frac{d}{dt}\mathbf{v}_\perp^{(1)} = \frac{q}{m}(\mathbf{v}_\perp^{(1)} \times \mathbf{B} + \mathbf{E}^{(1)}) \Rightarrow \mathbf{v}_\perp^{(0)} \cdot \frac{d}{dt}\mathbf{v}_\perp^{(1)} = \frac{q}{m}\mathbf{E}_\perp^{(1)} \cdot \mathbf{v}_\perp^{(0)}$$

- Term (ii) similarly, 1st order correction to the gyrophase θ, i.e., gyration speed is slightly nonuniform due to $\mathbf{E}_\perp^{(1)}$, $\rightarrow$Not of primary physical interest

- Now, we perform perturbation theory:

with

$$\Omega \gg \omega \sim k_\parallel v_\parallel,\ \frac{\omega}{\Omega} \sim \frac{e\delta\phi}{T} \ll 1,\ k_\parallel \ll k_\perp \sim \rho_i^{-1}$$

- Eq. (4)⇒

$$\underbrace{\Omega\frac{\partial f}{\partial\theta}}_{\text{Largest term}} + \left(\frac{\partial}{\partial t} + v_\parallel\hat{\mathbf{b}}\cdot\frac{\partial}{\partial\mathbf{R}} + c\frac{\mathbf{E}\times\mathbf{B}}{B^2}\cdot\frac{\partial}{\partial\mathbf{R}} - \frac{q}{m}\nabla_\parallel\phi\frac{\partial}{\partial v_\parallel} - \frac{q\Omega}{mB}\frac{\partial\phi}{\partial\theta}\frac{\partial}{\partial\mu}\right) f = 0 \tag{5}$$

Let $f = f^{(0)} + f^{(1)} + \cdots$, with expansion parameter $\delta \sim \frac{\omega}{\Omega} \sim \frac{k_\parallel v_\parallel}{\Omega} \sim \frac{|e|\phi}{T_e}$

- 0-th order $\Rightarrow \Omega\frac{\partial}{\partial\theta}f^{(0)} = 0 \Rightarrow f^{(0)}$ is independent ofθ,
  $\therefore f = \langle f\rangle + f_{AC}, \langle\cdots\rangle = \frac{1}{2\pi}\oint d\theta\{\cdots\}$gyrophase average
  with $f^{(0)} = \langle f\rangle, f^{(1)} = f_{AC} \ll f^{(0)} = \langle f\rangle$
- 1-st order $\Rightarrow$

$$\underbrace{\Omega\frac{\partial}{\partial\theta}f^{(1)}}_{\text{(a)}} + \left(\frac{\partial}{\partial t} + v_\parallel\hat{\mathbf{b}}\cdot\frac{\partial}{\partial\mathbf{R}} + c\frac{\mathbf{E}\times\mathbf{B}}{B^2}\cdot\frac{\partial}{\partial\mathbf{R}} - \frac{q}{m}\nabla_\parallel\phi\frac{\partial}{\partial v_\parallel} - \underbrace{\frac{q\Omega}{mB}\frac{\partial\phi}{\partial\theta}\frac{\partial}{\partial\mu}}_{\text{(b)}}\right) f^{(0)} = 0 \tag{6}$$

(a) and (b) can be combined into

$$\Omega\frac{\partial}{\partial\theta}\left[f_{AC} - \frac{q\phi}{mB}\frac{\partial\phi}{\partial\theta}\frac{\partial}{\partial\mu}\langle f\rangle\right]$$

- Taking gyro-phase average of Eq. (6): $\langle \cdots \rangle = \frac{1}{2\pi}\oint d\theta \cdots$

$$\langle \Omega \frac{\partial}{\partial \theta}\{\cdots\}\rangle = 0 \Rightarrow$$

$$\left[\frac{\partial}{\partial t} + v_\parallel \hat{\mathbf{b}} \cdot \frac{\partial}{\partial \mathbf{R}} + \frac{c}{B}\hat{\mathbf{b}} \times \nabla \langle \phi \rangle - \frac{q}{m}\hat{\mathbf{b}} \cdot \frac{\partial}{\partial \mathbf{R}}\langle \phi \rangle \frac{\partial}{\partial v_\parallel}\right] \langle f \rangle = 0 \tag{7}$$

Finally, the electrostatic NLGK vlasov equation in uniform B

- $\langle \phi \rangle$ contains the Finite Larmor Radius (FLR) effect!

although it's gyrophase-averaged

$$\phi(\mathbf{x}) = \phi(\mathbf{R} + \boldsymbol{\rho}) = \sum_{\mathbf{k}} \phi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} = \sum_{\mathbf{k}} \phi_{\mathbf{k}} e^{i\mathbf{k}_\perp \cdot \mathbf{R}} e^{ik_\perp \rho \sin\theta}$$

Fourier-Bessel Expansion:

$$e^{ik_\perp \rho \sin\theta} = \sum_n J_n(k_\perp \rho) e^{in\theta}$$

$$\langle e^{ik_\perp \rho \sin\theta} \rangle = \frac{1}{2\pi}\oint d\theta \sum_n J_n(k_\perp \rho) e^{in\theta} = J_0(k_\perp \rho)$$

$\therefore$

$$\langle \phi \rangle = \sum_{\mathbf{k}} J_0(k_\perp \rho) \phi_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{R}}$$

- Widespread Misconception: "Gyrokinetic Theory throws away the gyrophase-dependent information"

- Part of Reasons: Conventional (old-fashioned) derivation is rather opaque (much more complex in general geometry in nonuniform B)

  Illustration in this note is a bit "modernized" version than the original papers up to mid 80's.

  - Hard to identify the role or necessity of $\theta-$dependent information
  - Also, most attention was paid to the nonlinear GK-"Vlasov" Equations.

Gyrokinetic Poisson Equation

- Maxwell's Eqns are still fine!
  but was NOT written in g.c. coordinates $(\mathbf{R})$
- So we need to express $n_i(\mathbf{x})$ in terms of $\langle f \rangle(\mathbf{R}, \mathbf{v}_\parallel, \mu)$

$$(\mathbf{R},\ \mathbf{v}_\parallel,\ \mu,\ \theta) \Rightarrow (\mathbf{x},\ \mathbf{v})$$

"Pull-Back" Transformation for GK Maxwell's Eqn
(ES $\Rightarrow$ Poisson)

$$(\mathbf{x},\ \mathbf{v}) \Rightarrow (\mathbf{R},\ \mathbf{v}_\parallel,\ \mu,\ \theta)$$

"Push-Forward" Transformation for GK-Vlasov

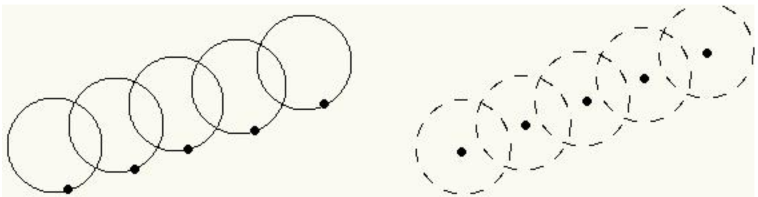
$$\nabla^2\phi = -4\pi e[n_i(\mathbf{x}) - n_e(\mathbf{x})]$$

- $n_i(\mathbf{x})$: typically obtained from GK Eqn
- $n_e(\mathbf{x})$: from adiabatic response for pure - ITG
  or from drift-kinetic or bounce-kinetic
  or from some other fluid eqns for more realistic case
  "GK" required for ETG

$$\begin{aligned} n_i(\mathbf{x}) &= \int d^3\mathbf{v} f_i(\mathbf{x},\ \mathbf{v}, t) \\ &= \int d^3\mathbf{x}' d^3\mathbf{v} f_i(\mathbf{x}',\ \mathbf{v})\delta(\mathbf{x}' - \mathbf{x}) \\ &= \int d^3\mathbf{R} d\mu dv_\parallel d\theta B f_i(\mathbf{R},\ \mu,\ v_\parallel, \theta)\delta(\mathbf{R} + \boldsymbol{\rho} - \mathbf{x}) \end{aligned} \tag{8}$$

not quite the same

$$\int d^3\mathbf{R} d\mu dv_\parallel B\langle f\rangle(\mathbf{R},\ \mu,\ v_\parallel)$$



Since

$$f_i(\mathbf{R},\ \mu,\ v_\parallel, \theta) = \langle f \rangle + f_{AC}(\mathbf{R},\ \mu,\ v_\parallel, \theta),$$

we need to know "f_{AC}" as well.
Back to Eq. (6):

$$\Omega \frac{\partial}{\partial \theta} \left[f_{AC} - \frac{q\phi}{mB} \frac{\partial \phi}{\partial \theta} \frac{\partial}{\partial \mu} \langle f \rangle \right] + \text{``} \frac{d}{dt} \langle f \rangle \text{''} = 0$$

and Eq. (7)

$$\left. \frac{d}{dt} \right|^{(0)} \langle f \rangle = 0$$

$\Rightarrow$

$$\Omega \frac{\partial}{\partial \theta} \left[f_{AC} - \frac{q\phi}{mB} \frac{\partial \phi}{\partial \theta} \frac{\partial}{\partial \mu} \langle f \rangle \right] + \left(\frac{d}{dt} - \left. \frac{d}{dt} \right|^{(0)} \right) \langle f \rangle = 0 \tag{9}$$

$$\frac{d}{dt} - \frac{d}{dt}^{(0)} \propto \text{``}\phi - \langle \phi \rangle\text{''}$$

integrating Eq. (9)

$$f_{AC}(\theta) \simeq \frac{q}{mB} (\phi - \langle \phi \rangle) \frac{\partial}{\partial \mu} \langle f \rangle \tag{10}$$

Polarization Density

Eq. (8) $\Rightarrow$

$$n_i(\mathbf{x}) = \underbrace{\int d^3\mathbf{R}\, d\mu\, dv_\parallel\, d\theta\, B \langle f \rangle \delta(\mathbf{R} + \boldsymbol{\rho} - \mathbf{x})}_{n_{i,gc}(\mathbf{x})}$$

$$+ \underbrace{\int d^3\mathbf{R}\, d\mu\, dv_\parallel\, d\theta\, B \frac{q}{mB} \left(\phi - \langle \phi \rangle\right) \frac{\partial \langle f \rangle}{\partial \mu} \delta(\mathbf{R} + \boldsymbol{\rho} - \mathbf{x})}_{n_{pol}(\mathbf{x})}$$

- $n_{i,gc}(\mathbf{x})$: G.C. density at particle position
- $n_{pol}(\mathbf{x})$: Polarization Density, one can evaluate exactly for $\langle f \rangle \propto e^{-\mu B/T}$, i.e.,"Maxwellian in $\mu \propto v_\perp^2$"

➔ $(1-\Gamma_0)\delta\phi$

➔ For long wavelength ; $\propto \vec{\nabla}_\perp \bullet \left(\frac{n}{B^2} \vec{\nabla}_\perp \delta\phi \right)$

Nonlinear Gyrokinetics for Large Scale Computation

• Direct simulation of actual size fusion plasmas in realistic geometry using the primitive nonlinear plasma equations (Vlasov-Maxwell), is far beyond the computational capability of foreseeable future.

• For turbulence problems in fusion plasmas, the temporal scales fluctuations much longer than the period of a charged particle's cyclotron motion, while the spatial scales and gyro-orbits are much smaller than the macroscopic length scales: → details of the charged particle's gyration motion are not of physical interest → Develop reduced dynamical equations which capture the essential features

• After decoupling of gyro-motion, gyrokinetic equation describes evolution of gyro-center distribution function, independent of the gyro-phase, θ, defined over a five-dimensional phase space ($\mathbf{R}$, $v_{||}$, μ). → save enormous amounts of computing time by having a time step greater than the gyro-period, and by reducing the number of dynamical variables.

• In gyrokinetic approach, gyro-phase is an ignorable coordinate, magnitude of the perpendicular velocity enters as a parameter in terms of an adiabatic invariant μ

• Nonlinear gyrokinetic equations are now widely used in turbulence simulations.

Modern Nonlinear Gyrokinetics

- Starting from the original Vlasov-Maxwell system (6D), pursue **"Reduction of dimensionality"** for both computational and analytic feasibility.

- Keep intact the underlying symmetry/conservation of the original system.

- Perturbation analysis consists of near-identity coordinate transformation which **"decouples"** the gyration from the slower dynamics of interest in the single particle Lagrangian, rather than a direct "gyro-phase average" of Vlasov equation.

- This procedure is reversible:

The gyro-phase dependent information can be recovered when it is needed.

Phase Space Lagrangian Derivation of Nonlinear Gyrokinetics

[since Hahm, PF **31**, 2670 '88, followed by Brizard, Sugama,…]

- **Conservations Laws are Satisfied.**
- Various expansion parameters appear at different stages

→Flexibility in variations of ordering for specific application

- Guiding center drift calculations in equilibrium field **B**:

Expansion in $\delta_B = \rho_i / L_B \sim \rho_i / R$.

- Perturbative analysis consists of near-identity transformations to new variables which remove the gyro-phase dependence in perturbed fields $\delta\mathbf{A}(\mathbf{x})$, $\delta\phi(\mathbf{x})$ where $\mathbf{x} = \mathbf{R} + \rho$:

Expansion in $\varepsilon_\phi = e[\delta\phi - (v_{||}/c)\delta A_{||}]/T_e \sim \delta B_{||}/B_0$.

- Derivation more transparent, less amount of algebra

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Hierarchy of Nonlinear Governing Equations

Nonlinear equations: From fundamental, primitive to reduced, simplified	Steps for reduction	Physics lost due to reduction
Vlasov-Klimontovich equation [Klimontovich 67]	Remove high frequency terms ($\geq \omega_{ci}$)	
Gyrokinetic Equation: Conservative [Dubin 83, Hahm 88a,b, Brizard 89, 06]	Neglect velocity space nonlinearity	High frequency phenomena
Gyrokinetic Equation: Conventional [Frieman 82]	Take moments in velocity space	Conservation of energy between particles and fields, of phase-space volume, nonlinear trapping of particles along $\boldsymbol{B}$.
Gyrofluid Equation [Beer 96, Dorland 93]	Expansion in finite Larmor radius terms; Ordering for collisional plasmas	Some nonlinear kinetic effects including inelastic Compton scattering [Mattor 92], accuracy in damping rates of zonal flow [Rosenbluth 96] and damped mode [Sugama97]
Fluid Equations [Yagi 94, Scott 97, Zeiler 97, Xu 02, Simakov 05]		Most kinetic effects associated with long mean free paths and finite size orbits.

from Diamond-Itoh-Itoh-Hahm : PPCF **47**, R35, (2005)

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Shear-Alfvén Continuum in Sheared Magnetic Field

- When driving is weak, shear-Alfvén wave DR. $\omega^2 = k_\parallel^2 \mathrm{v_A}^2$

  ➔In sheared magnetic field, with $k_\parallel = \dfrac{nq(r)-m}{qR} \cong \dfrac{k_\theta}{L_s}(x - x_{m,n})$

- For given n, m, linear D.R. is satisfied at least one radial position for any reasonably small values in $k_\parallel$

  ➔ as $k_\parallel(x)$ is varied as a function of x , ω assumes “continuum” of values , rather than an “eigenvalue” (discretized)

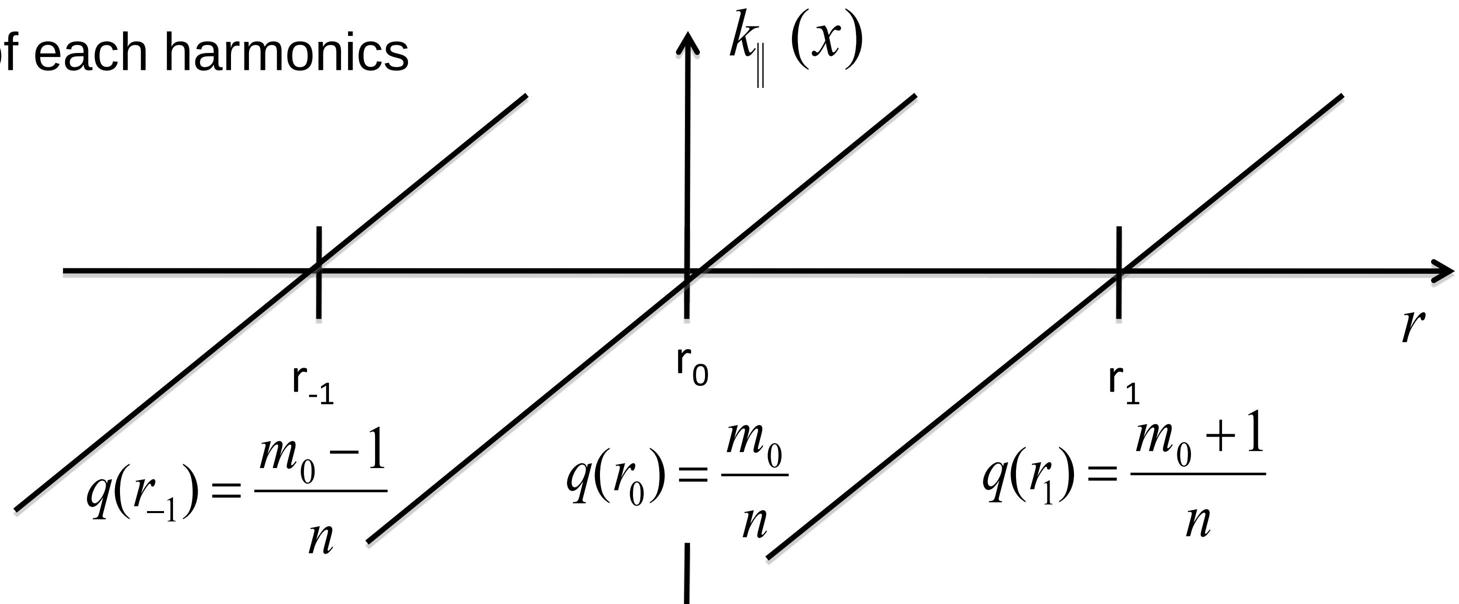
  ➔ Alfvén continuum ➔ initial wave packet will phase-mix and decay algebraically in time.

- Then what’s the consequence of toroidal geometry?

  i.e., coupling between neighboring poloidal harmonics

Linear Coupling of Poloidal Harmonics

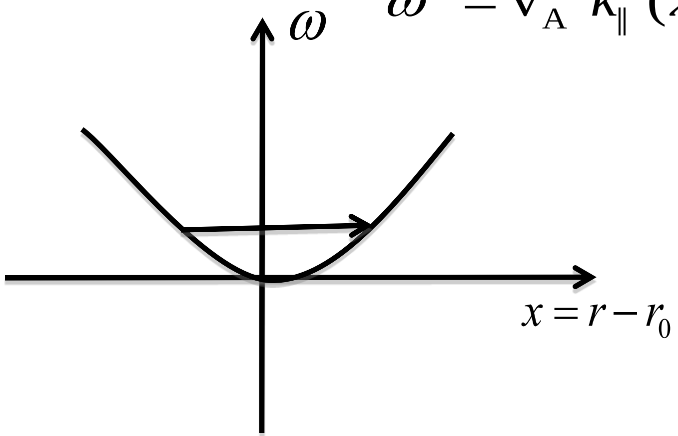
- $k_\parallel$ of each harmonics



- Shear-Alfvén Continuum of each poloidal harmonics (in Slab)

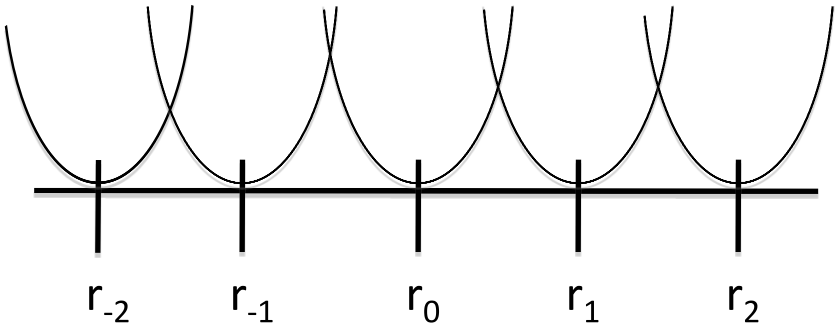
$$\omega^2 = \mathrm{v_A}^2 k_\parallel^2(x)$$

For $\forall \omega$, dispersion relation satisfied at one x.

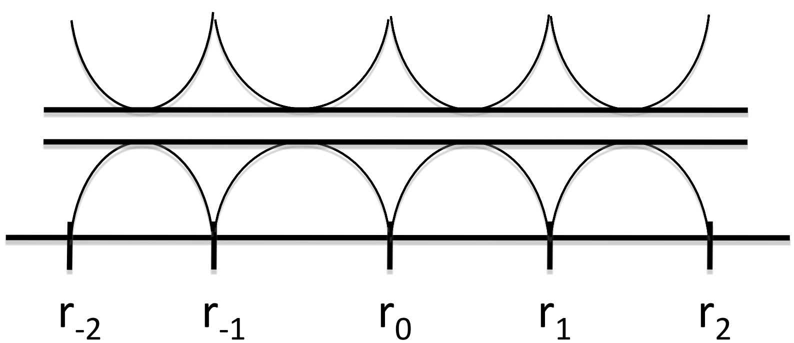


Toroidicity-Induced Alfvén GAP

$$\omega^2 = k_\parallel^2 v_A^2, \text{ with } k_\parallel \propto \frac{k_\theta}{L_s}(r - r_m)$$



$$\omega^2 = \omega_A^2(r)$$



Each harmonics considered in sheared slab

$$v_A^2 \propto B^2 = \text{const}$$

$$B = \frac{B_0}{1 + \frac{r}{R_0}\cos\theta}$$ induces toroidal GAP

→ (involves solving Mathieu Equation)

Continuum Dispersion Relation Not Satisfied for

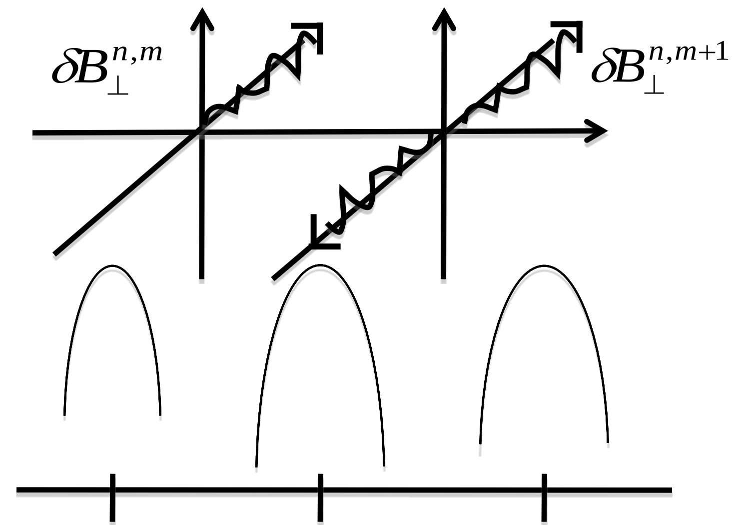
$$\frac{v_A}{2qR}(1-\varepsilon) < \omega < \frac{v_A}{2qR}(1+\varepsilon)$$

Toroidal Alfvén Eigen modes

- At the midpoint between two adjacent rational surfaces
  $k_{\parallel} = \frac{1}{2qR}$ ~ GAP occurs near $\omega \simeq \frac{v_A}{2qR}$
- "Standing wave formation" from

superposition of

$\delta B_{\perp}{}^{n,m} e^{;\mathrm{co-prop}}$ and $\delta B_{\perp}{}^{n,m+1} e^{;\mathrm{counter-prop}}$



- This "TAE" modes can be excited via

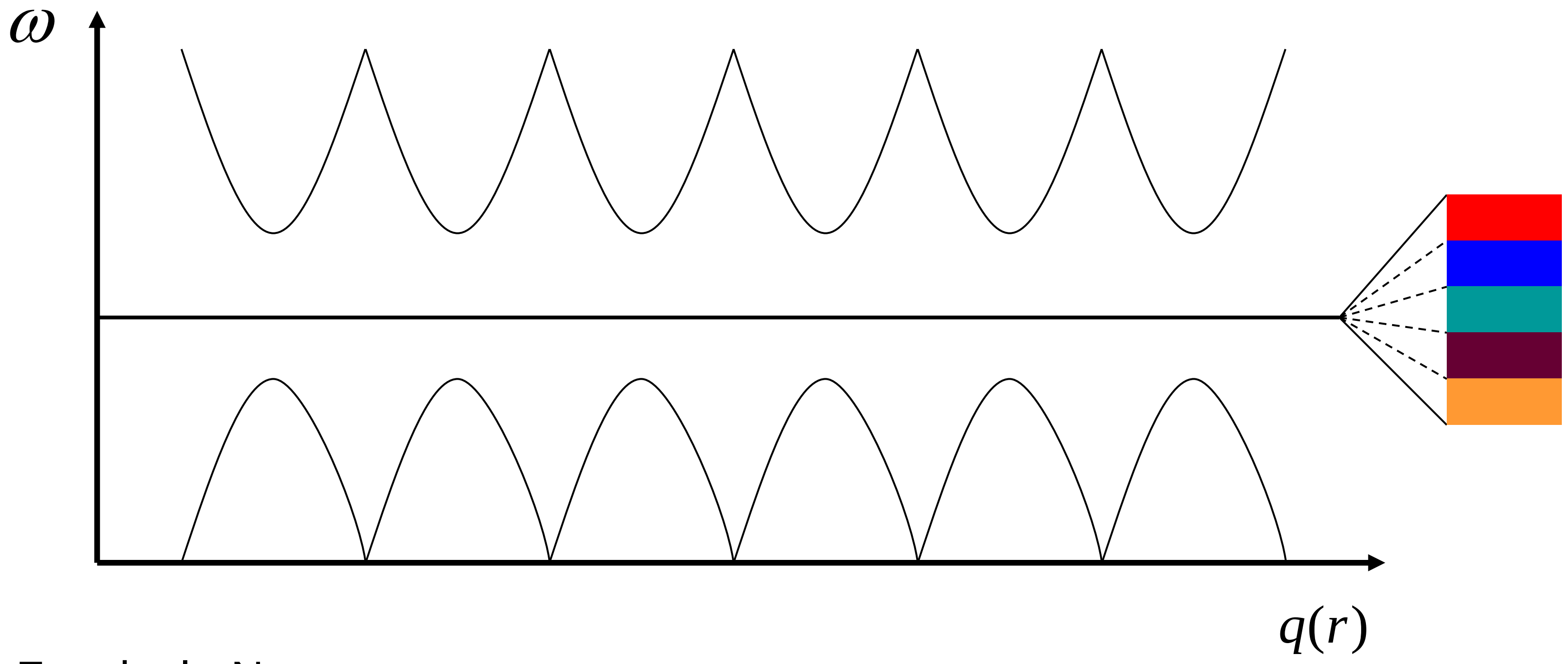
resonance with energetic ions.

- … AE zoo accommodates TAE, BAE, GAE, CAE, HAE, EAE, LSAE, RSAE, …, and

Nonconventional AE !

Translational Invariance

- If $\frac{v_A}{qR_0}$ is uniform in r, $\sim N(q(a)-q(0))$ modes have the same eigenfrequency (degenerate).



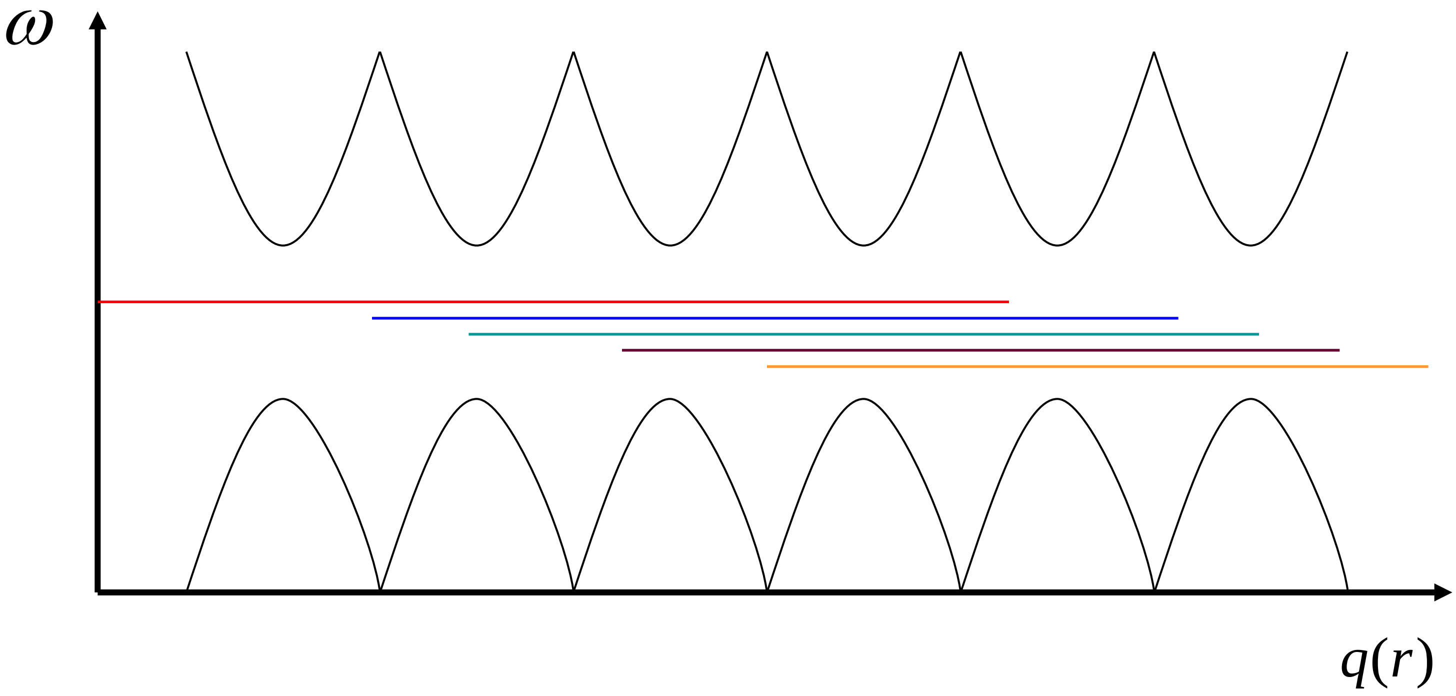
For single-N,

Quasi-Translational Invariance

- With equilibrium variation, degeneracy is broken.

Each TAE's has slightly different eigenfrequency.

"High-N TAE" still contains many poloidal harmonics.



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Nonlinear Saturation Mechanism for High-N TAE

" Ion Compton Scattering "

TAE ω

TAE ω'

$\delta J_{\perp} \times \delta B_{\perp}$ Nonlinearity

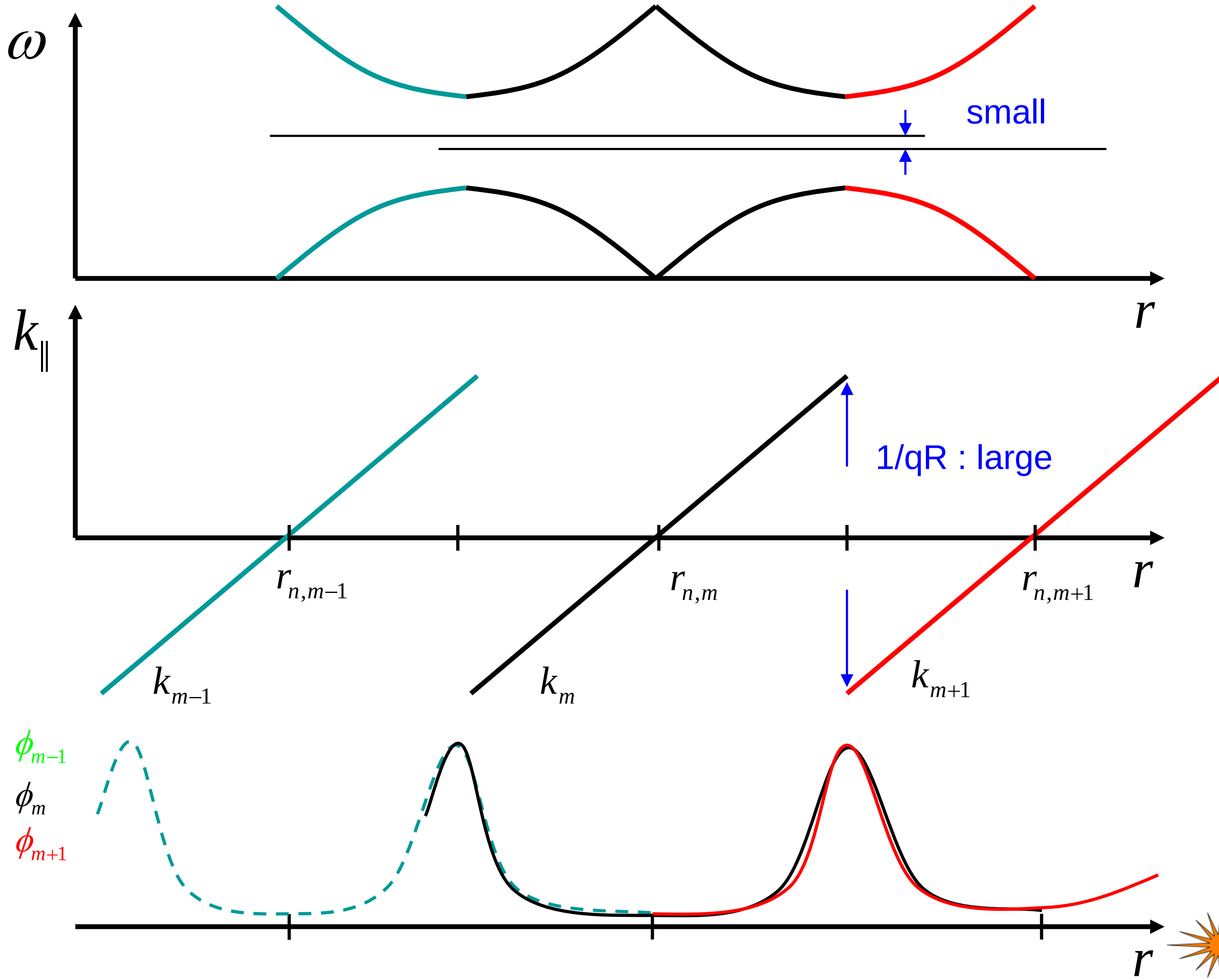
" BEAT WAVE "

$$\omega'' = \omega - \omega'$$
$$k''_{\parallel} = k_{\parallel} - k'_{\parallel}$$

- If $\dfrac{\omega''}{k''_{\parallel}} \sim \mathrm{v}_{Ti} << \mathrm{v}_{A}$ (Bulk), via Compton Scattering, fluctuation energy is transferred to lower frequency mode, eventually absorbed by linearly stable mode near lower continuum.

→ " Look for Nonlinear Coupling Channel " via beat wave with low phase velocity which gives $\dfrac{\omega''}{k''_{\parallel}} \sim \mathrm{v}_{Ti}$!

Generation of Low Phase Velocity Beat Wave



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Nonlinear Interaction of TAE's → Sound Wave-like Density Fluctuation

	Test mode (TAE)	Background mode (TAE)	Beat wave (Sound wave)
Wave Vector	$\vec{k}$	$\vec{k}'$	$\vec{k}'' = \vec{k} - \vec{k}'$
Frequency	ω	ω'	$\omega'' = \omega - \omega' \lesssim \varepsilon\omega$ small
$k_\parallel$ **@ gap**	$\frac{1}{2qR}$	$-\frac{1}{2qR}$	"$\frac{1}{qR}$" large
• Single – n :			
Toroidal Mode Number	n	n	0
Poloidal Mode Number	m	$m+1$	-1
• For multi – n :			
Toroidal Mode Number		n'	$n - n' << n$
Poloidal Mode Number		$m'+1$	$m - m' - 1 << m$

* This talk : $10^1 \le N << 10^2$

Weak Turbulence Nonlinear Analysis *[Hahm&Chen, PRL '95]*

- 3rd Order Perturbation Theory : $\frac{\gamma_L}{\omega_A} << 1$

- 1st order : Test TAE ($\vec{k}$) ideal MHD

$$\frac{\partial}{\partial t}\psi_{\vec{k}}^{(1)} = -\hat{\mathbf{n}}\cdot\nabla\phi_{\vec{k}}^{(1)}$$

$$\frac{\partial}{\partial t}\nabla_\perp^2\phi_{\vec{k}}^{(1)} = -\mathrm{v}_A^2\hat{\mathbf{n}}\cdot\nabla\nabla_\perp^2\psi_{\vec{k}}^{(1)}$$

- 2nd order : Nonlinear Interaction of two TAE's, $\begin{pmatrix}\vec{k}\\ \vec{k}'\end{pmatrix}$.

  → Sound-wave-like density fluctuation, $\vec{k}''$.

$$\left(\frac{\partial}{\partial t} + \mathrm{v}_\parallel\hat{\mathbf{n}}\cdot\nabla\right)\delta f_{\vec{k}''}^{(2)} = \left\{\left(\frac{\delta\vec{B}}{B_0}\cdot\frac{\partial}{\partial t}\vec{\mathrm{v}}_E\right)_{\vec{k}''} + \frac{|e|}{M_i}\hat{\mathbf{n}}\cdot\nabla\phi_{\vec{k}''}\right\}\frac{\partial f_0}{\partial \mathrm{v}_\parallel}$$

$\delta J_\perp \times \delta B_\perp$ NL

" $\omega'' = k_\parallel''\mathrm{v}_\parallel$ Resonance important."

- 3rd order : Nonlinear Evolution of Test TAE ($\vec{k}$),

  in the presence of density fluctuation $\delta n_{\vec{k}''}^{(2)}$

  and other TAE's $\phi_{\vec{k}'}^{(1)}$.

$$\mathrm{v}_A^2\hat{\mathbf{n}}\cdot\nabla\nabla_\perp^2\psi_{\vec{k}} + \frac{\partial}{\partial t}\nabla_\perp^2\phi_{\vec{k}} + \sum_{\vec{k}'}\nabla\cdot\left(\frac{\delta n_{\vec{k}''}^{(2)}}{n_0}\frac{\partial}{\partial t}\nabla_\perp\phi_{\vec{k}'}^{(1)}\right) = 0.$$

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Nonlinear Evolution of TAE in the presence of density fluctuation

$$v_{A0}^2\left(\hat{n}\cdot\nabla\nabla_\perp^2\psi_{\vec{k}}+\sum_{\vec{k}'}\nabla\psi_{\vec{k}'}\times\hat{n}\cdot\nabla\nabla_\perp^2\psi_{\vec{k}''}\right)+\frac{\partial}{\partial t}\nabla_\perp^2\phi_{\vec{k}}+\sum_{\vec{k}'}\nabla\phi_{\vec{k}'}\times\hat{n}\cdot\nabla\nabla_\perp^2\phi_{\vec{k}''}+\sum_{\vec{k}'}\nabla\cdot\left(\frac{\delta n_{\vec{k}''}^{(2)}}{n_0}\right)\frac{\partial}{\partial t}\nabla_\perp\phi_{\vec{k}'}=0$$

- Recall; Vorticity Equation is $\vec{\nabla}_\perp\cdot\delta\vec{J}_\perp^{\,\mathrm{pol}}+\nabla_\parallel\cdot\delta J_\parallel=0$

  where "$\delta\vec{J}_\perp^{\,\mathrm{pol}}=\frac{1}{B_0}\hat{n}\times\rho\frac{d\vec{v}_E}{dt}$" depends on "number density."

- Other nonlinearities are subdominant for $k'_\perp k''_\perp\rho_s^2<<\frac{\omega'}{\Omega_{ci}}$

---

* Multiplying $\phi_{\vec{k}}^*$, take imaginary part of the radial average, we get

$$\frac{\partial}{\partial t}I_{\vec{k}}=\gamma_L(\vec{k})I_{\vec{k}}-\sum_{\vec{k}'}M_{\vec{k},\vec{k}'}I_{\vec{k}'}I_{\vec{k}}$$

$$I_{\vec{k}}\equiv\left\langle\left|\nabla_\perp\phi_{\vec{k}}\right|^2\right\rangle,\quad M_{\vec{k},\vec{k}'}\equiv\frac{\omega'}{2}\frac{\chi_e^2\,\mathrm{Im}\,\chi_i}{\left|\chi_e+\chi_i\right|^2}\frac{M_i}{B_0}$$

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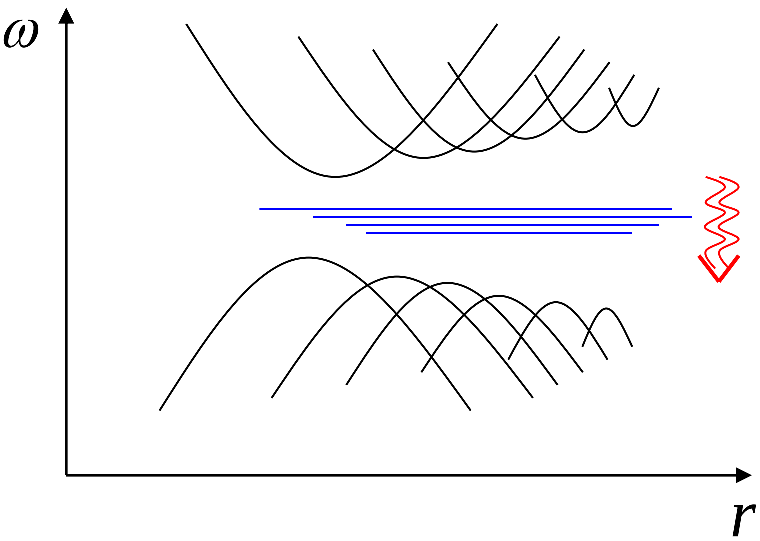
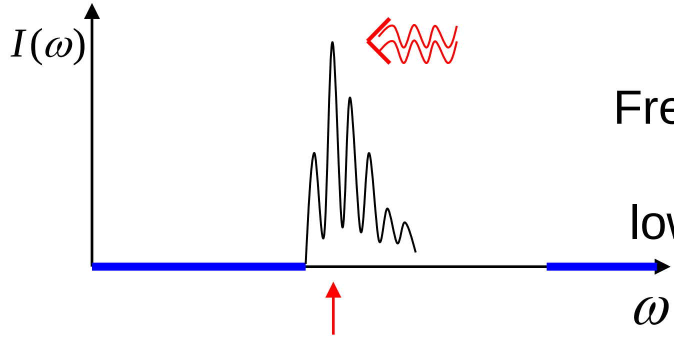
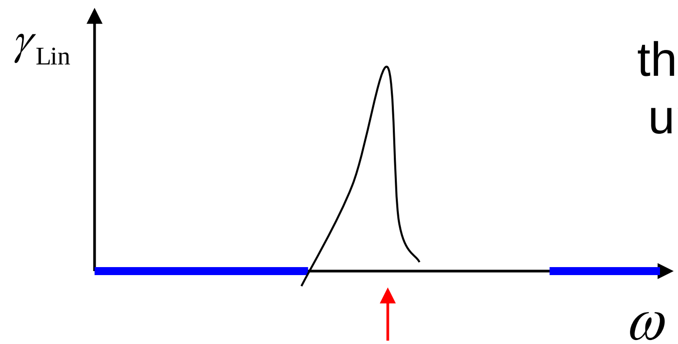
“ Frequency Chirping from Ion Compton Scattering ”





Frequency chirps
towards
lower continuum

from
the linearly most
unstable mode.



Fluctuation Energy is transferred to
Lower Frequency due to Ion Compton Scattering.

At Nonlinear Saturation

$$“\left(\frac{\delta B_r}{B_\varphi}\right)^2 \simeq \frac{1}{4\pi}\left(1+\frac{T_e}{T_i}\right)^2\left(\frac{\bar{\gamma}_L}{\omega_A}\right)\left(\frac{r}{R_0}\right)^4”$$

- Magnitude :

$$\frac{\bar{\gamma}_L}{\omega_A} < \frac{\bar{\gamma}_A^{\text{Max}}}{\omega_A} \lesssim 10^{-2}$$

$$\frac{r}{R_0} \sim 10^{-1} \quad \Rightarrow \quad \frac{\delta B_r}{B_\varphi} \lesssim 10^{-3}$$

- Scaling :

$$\frac{\delta B_r}{B_\phi} \propto \left(\frac{\bar{\gamma}_L}{\omega_A}\right)^{\frac{1}{2}}$$

: weak turbulence theory

(⟺ $\delta B_r \propto \gamma_L^2$; à la single wave trapping) Berk, Breizman, Fu, W. Park(not H. Park), Wu, White, Rosenbluth …

Our Mechanism :

- More relevant for High-N Multi-mode Overlapping Case. Also for stronger drive.

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Single Wave Trapping (Berk-Breizmann et al.)

- Resonant particles are trapped in the potential well produced by TAE.
- The potential well will last an auto-correlation time, ω_{ac}^{-1}.
- A trapped particle will transverse a closed orbit in, ω_b^{-1}, mixing hot-particle distribution function.

$\rightarrow$ Require : $\underline{\omega_b > \omega_{ac}}$

Nonlinear Saturation : $\omega_b \sim \gamma_{\text{Lin}}$

- It also requires a well-defined potential well in space,

$$\underline{\Delta X_{\text{particle, excursion}} < \Delta X_{N,M}}$$

,where $\Delta X_{N,M}$: distance between neighboring rational surfaces.

Weak Turbulence Theory

Requires

$$\omega_b < \omega_{ac}$$

and

$$\Delta X_{\text{ptl. exc.}} > \Delta X_{N,M}$$

Chirikov Criterion

Validity Regimes

* For Quasi-Linear Theory (Weak Turbulence Expansion)

  • $\Delta X_{\text{ptl. exc.}} > \Delta X_{N,M}$

$$\Rightarrow \quad \frac{\delta B_r}{B_\phi} > \frac{1}{8q^4\hat{s}}\left(\frac{r_0^2}{R_0\rho_\alpha}\right)\cdot\frac{1}{N^4}$$

  • $\omega_b < \omega_{ac}$

$$\Rightarrow \quad \frac{\delta B_r}{B_\phi} > \frac{1}{8q^4\hat{s}}\left(\frac{r_0^2}{R_0\rho_\alpha}\right)\cdot\frac{4\varepsilon^2\hat{s}^2}{N^2}$$

* For Single Wave Trapping

$\Delta X_{\text{ptl. exc.}} < \Delta X_{N,M}$, (Then, $\omega_b > \omega_{ac} \rightarrow 0$)

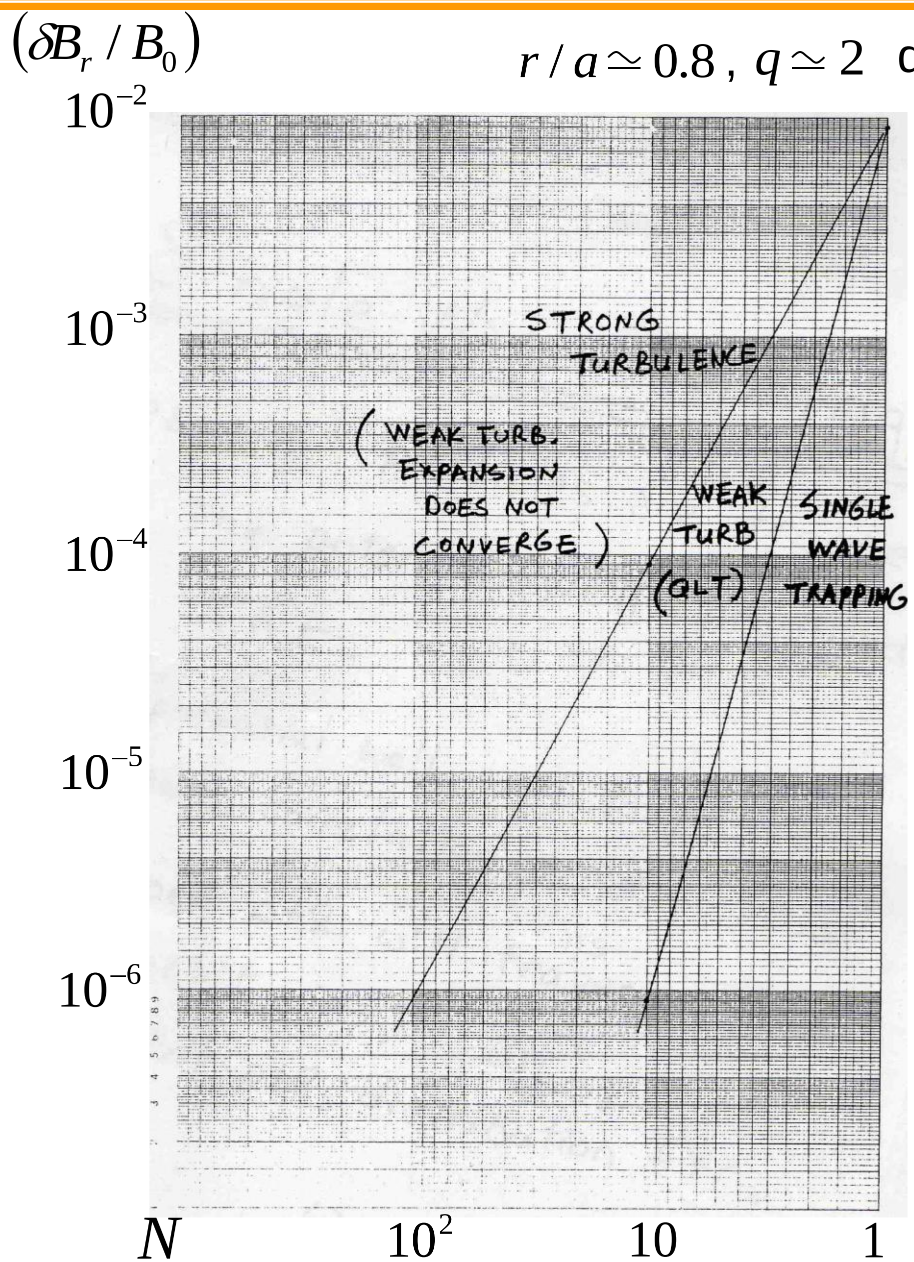
PLOT : in $\left(\frac{\delta B_r}{B_\phi}, N\right)$ space

Approximate equilibrium parameters for ITER PRETOR.

[Candy, Rosenbluth, NF '95]

Validity Regimes For ITER PRETOR Parameters

$r/a \simeq 0.8$, $q \simeq 2$ dense mode packing



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Incompressible MHD Turbulence

Tutorial at Theorié-Fest, Aix-en-Provence, 200? by P.H. Diamond

connects :

- Weak Turbulence Kinetic Theory : Sagdeev & Galeev '67
- Scaling Derivation : PHD & Craddock: Comments in Plasma Phys '90
  Lazarian & Vishniac: Ap. J. '99

and

- Lengthy Crank : Goldreich & Sridhar: Ap. J. '95

Common Theme :

Nonlinear Interaction of Low Frequency Beat Wave
with
Particles or Eddys

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	Weak Turbulence Theory of Incompressible MHD Turb.	WTT of Toroidal Alfvén Eigenmodes in Tokamaks
High Freq. Shear-Alfvén Waves interact nonlinearly with	Eddys	Thermal Ions
allowed by	Low Freq. Beat Wave produced by Counter-Prop^n	Low Frequency Beat Wave produced by "standing" TAEs (formed linearly by Counter-Prop^n of neighboring harmonics)
resulting in	Alfvén Effect (down by $\frac{\Delta\omega_k}{k_\parallel \mathrm{v}_A}$) and Scale-dependent Anisotropy (Intermediate Turbulence : G&S Ap. J. '97 $\simeq$ Anisotropic I-K '65)	Down-shift of Frequency $k_\parallel$ determined by equilibrium $\vec{B}$ geometry $k_\perp$ by linear drive (energetic particles)
Theory breaks down	when $k_\parallel \mathrm{v}_A = \mathrm{v}_\perp / \ell_\perp$ for small scales first	with non-overlapping island (violation of Chirikov Criterion) in phase-space, for large scales first
turning into	Critically Balanced Cascade (G&S '95 '97)	Single-Wave Trapping → Hole-Clump Pair in phase-space. Berk-Breizmann Paradigm

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Conclusions

Alfvén waves have been studied from different angles.

- MHD Turbulence Community : Fixation with k-spectra

  …

- MFE Theory Community : Fixation with linear instability zoology

  TAE, GAE, RSAE, LSAE, …

- Nonlinear Gyrokinetic Theory-based

Extensions may bridge some gaps.