DIFFUSIVE PROPAGATION OF UHECR

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SYROVATSKY (1959) SOLUTION OF DIFFUSION EQUATION

Equation for a single source:

$$\frac{\partial}{\partial t}n_p(E,\vec{r},t) - \mathrm{div}\,[D(E,\vec{r},t)\nabla n_p] - \frac{\partial}{\partial E}\,[b(E,\vec{r},t)n_p] = Q(E,\vec{r},t)\delta^3(\vec{r}-\vec{r}_g).$$

solution was obtained by exclusive method introducing the **Syrovatsky variables**

$$\lambda(E,E_g) = \int_E^{E_g} d\varepsilon \frac{D(\varepsilon)}{b(\varepsilon)}, \qquad \tau(E,E_g) = \int_E^{E_g} \frac{d\varepsilon}{b(\varepsilon)}.$$

This method is valid when $D(E)$, $b(E)$, Q(E) do not depend on time.

The Syrovatsky solution:

$$n_p(E,r) = \frac{1}{b(E)}\int_E^{\infty} dE_g Q(E_g)\frac{exp\left[-r^2/4\lambda(E,E_g)\right]}{\left[4\pi\lambda(E,E_g)\right]^{3/2}}.$$

SYROVATSKY SOLUTION AND PROPAGATION THEOREM

The Syrovatsky solution obeys the **propagation theorem** (Aloisio and VB 2004):

FOR UNIFORM DISTRIBUTION OF SOURCES WITH SEPARATION d MUCH LESS THAN CHARACTERISTIC LENGTHS OF PROPAGATION, SUCH AS $l_{\rm att}(E)$ and $l_{\rm diff}(E)$, **THE DIFFUSE SPECTRUM OF UHECR HAS AN UNIVERSAL (STANDARD) FORM INDEPENDENT OF MODE OF PROPAGATION .**

when $d \rightarrow 0$ solution for any mode of propagation tends to **universal spectrum**, which for homogeneous distribution of sources can be calculated from conservation of number of particles in the comoving volume $n_P(E)dE = \int dtq[E_g(t),t]dE_g$, where q is the production rate per unit comoving volume.

$$J_{\rm univ}(E) = \frac{c}{4\pi}\frac{\mathcal{L}_0(\gamma_g-2)}{E_{\rm min}^2}\int_0^{z_{\rm max}} dz \left|\frac{dt}{dz}\right| (1+z)^m \left(\frac{E_g(E,z)}{E_{\rm min}}\right)^{-\gamma_g} \frac{dE_g}{dE},$$

where $\mathcal{L}_0$ is emissivity and m describes evolution.

CALCULATION OF THE DIFFUSE FLUX

We calculate diffuse spectrum for sources located in vertices of cubic lattice

$$J_p(E) = \frac{c}{4\pi}\frac{1}{b(E)}\sum_i \int_E^{E_{max}} dE_g Q(E_g) \frac{exp\left[-r_i^2/4\lambda(E,E_g)\right]}{\left(4\pi\lambda(E,E_g)\right)^{3/2}}.$$

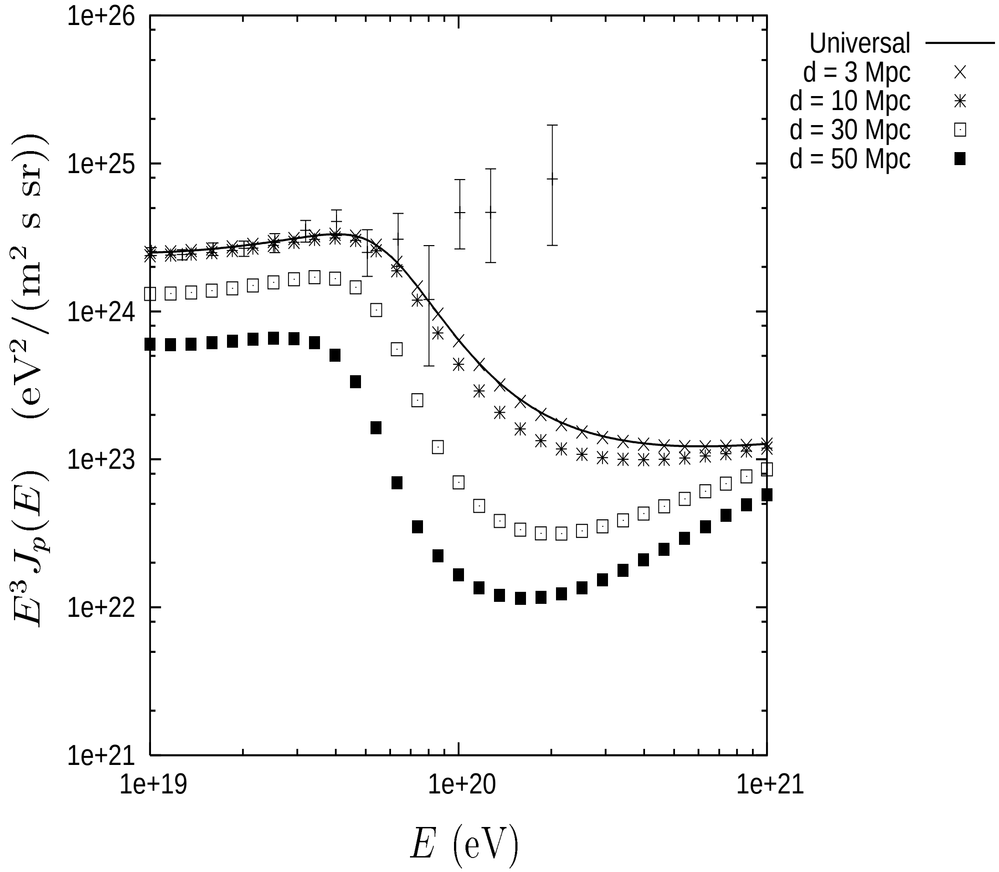
The diffusion coefficient $D(E)$ is needed for calculation of $\lambda(E, E_g)$.

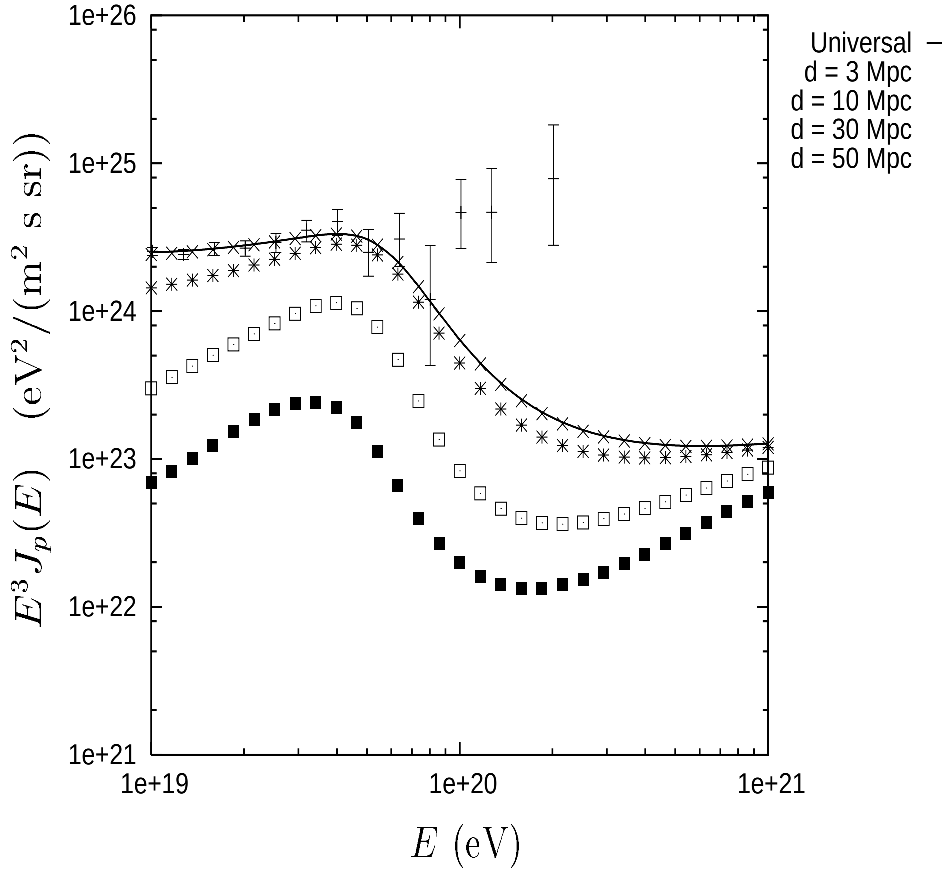
We assume magnetic turbulent plasma described as ensemble of MHD waves. Diffusion occurs due to resonant scattering on MHD waves. Magnetic turbulence has the basic (largest) scale l_c with magnetic field B_c.

It determines the critical energy E_c by relation $r_L(E_c) = l_c$.

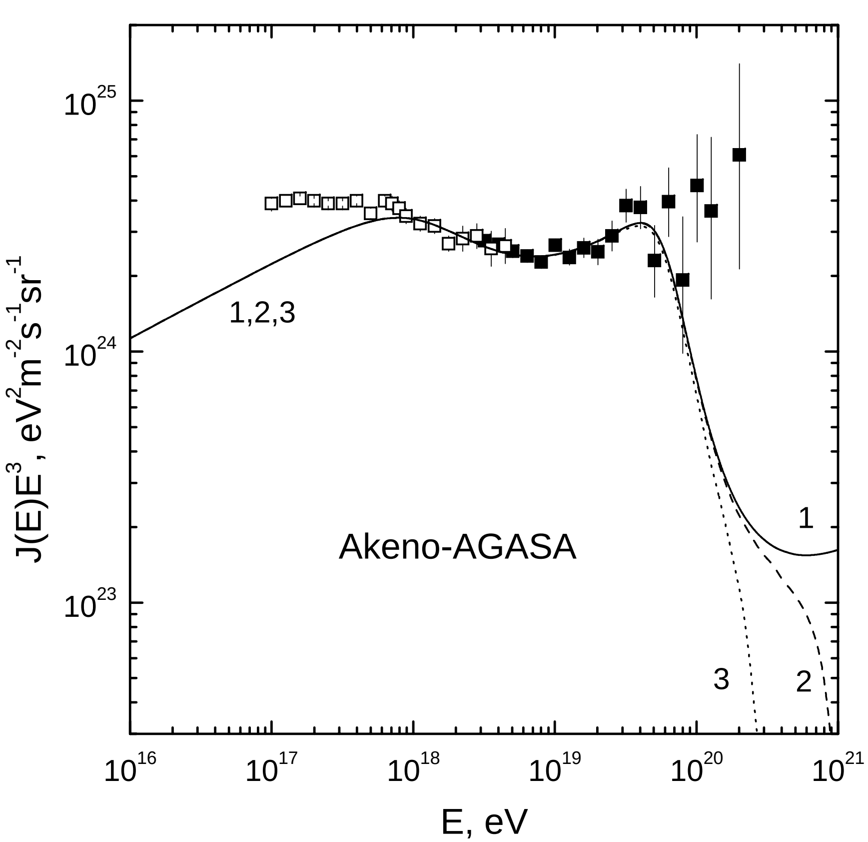
At $E \gg E_c$ $\;D(E) \approx cr_L^2/l_c \sim E^2$ for any spectrum of turbulence.
At $E \ll E_c$ $\;D(E)$ is determined by spectrum of turbulence,
e.g. $D(E) \sim E^{1/3}$ for the Kolmogorov spectrum.
Another option is the Bohm diffusion $D(E) = cr_L(E) \sim E$.

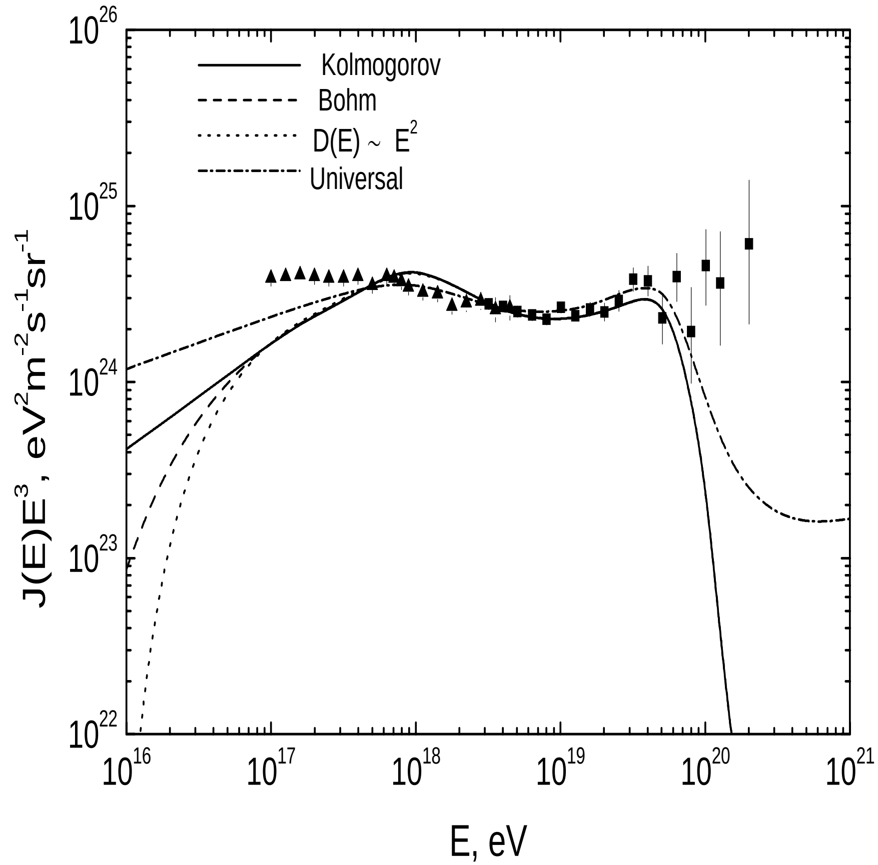
CONVERSION OF DIFFUSIVE SPECTRUM TO UNIVERSAL SPECTRUM





DIFFUSION at LOW-ENERGY END of UHECR





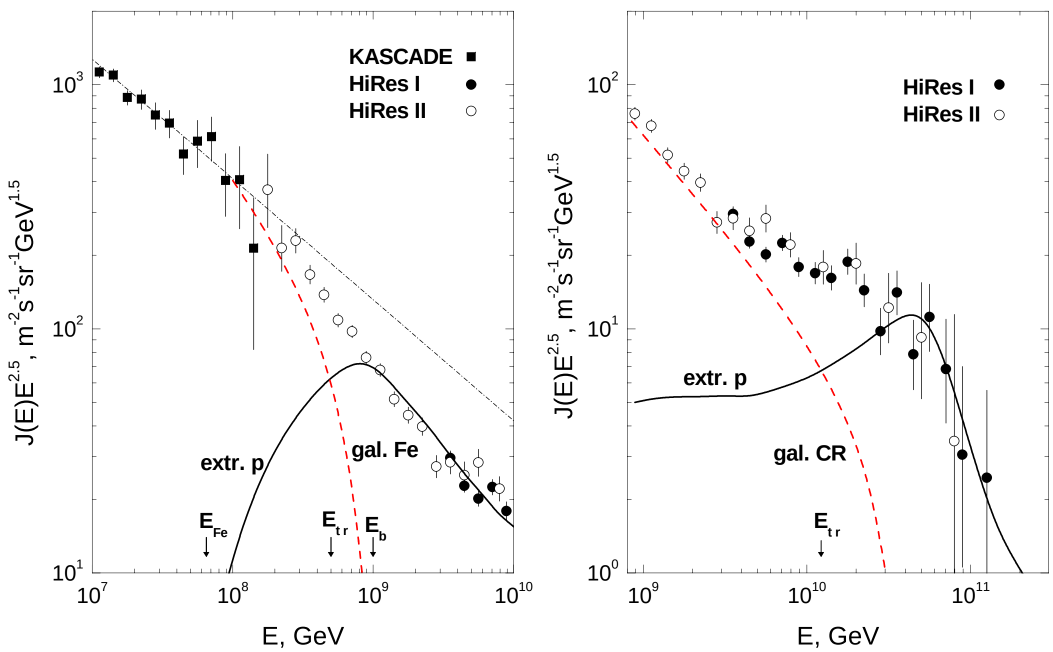
. **rectilinear propagation** **diffusive propagation**

The low-energy ’diffusive cutoff’ at $E_b = 1 \times 10^{18}$ eV is universal and valid for all propagation modes. It is determined by fundamental energy $E_{\rm eq} = 2 \times 10^{18}$ eV, where pair-production and adiabatic energy losses become equal. The spectrum at $E < E_b$ depends on mode of propagation, e.g. rectilinear, Bohm or Kolmogorov diffusion. The low-energy ’cutoff’ provides transition from extragalactic to galactic CR.

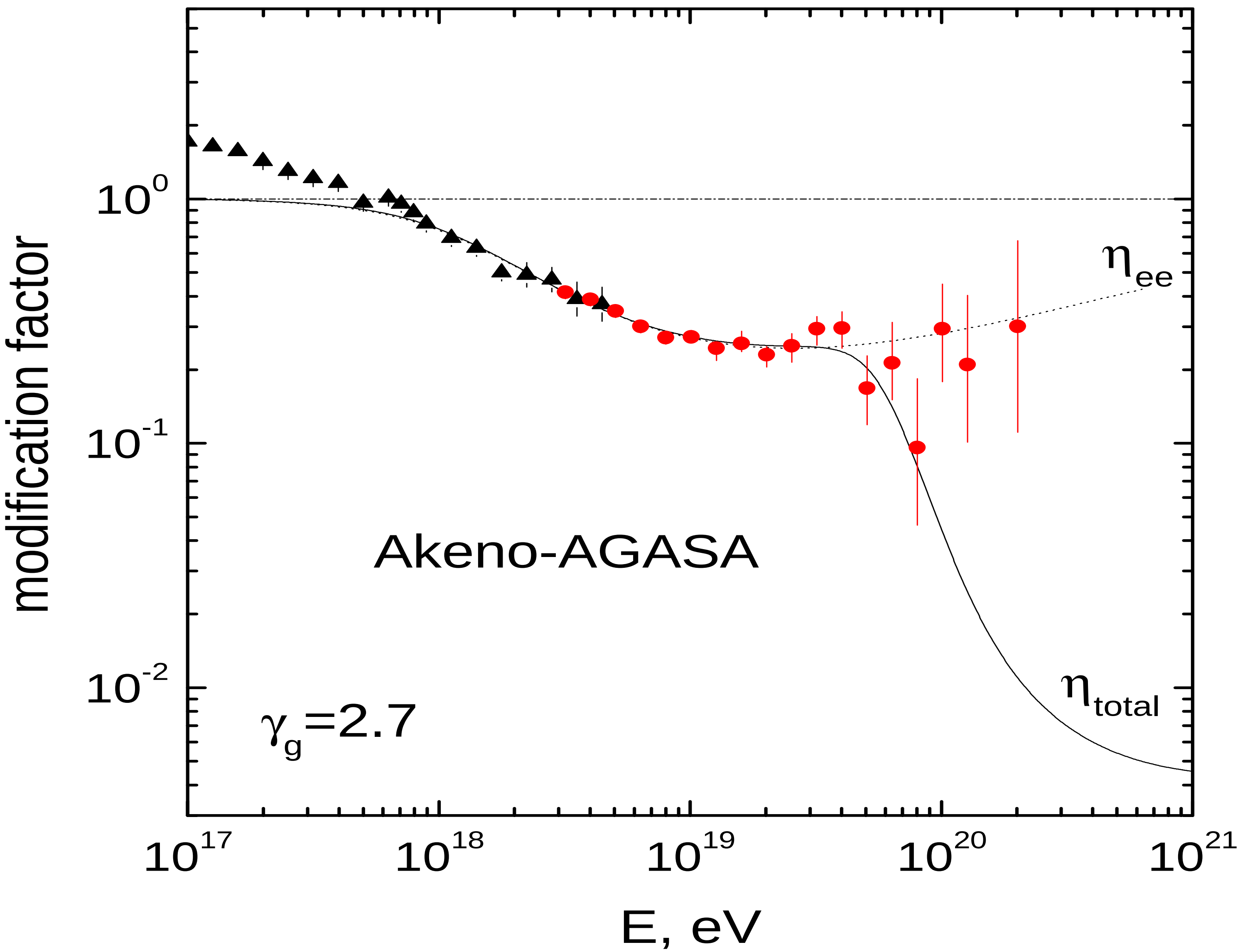
SECOND-KNEE and ANKLE TRANSITIONS

Transition occurs at $E_{\rm tr} < E_b = 1 \times 10^{18}$ eV, i.e. at **second knee**. This transition agrees well with rigidity-dependent position of **iron knee** $E_{\rm Fe} = ZE_p \approx 6.5 \times 10^{16}$ eV, where $E_p \approx 2.5 \times 10^{15}$ eV if proton knee. The galactic acceleration maximum $E_{\rm Fe}^{\rm max} \lesssim 10^{18}$ eV is satisfied. The predicted feature of extragalactic proton interaction with CMB at $E \geq 1 \times 10^{18}$ eV (**dip**) is well confirmed.

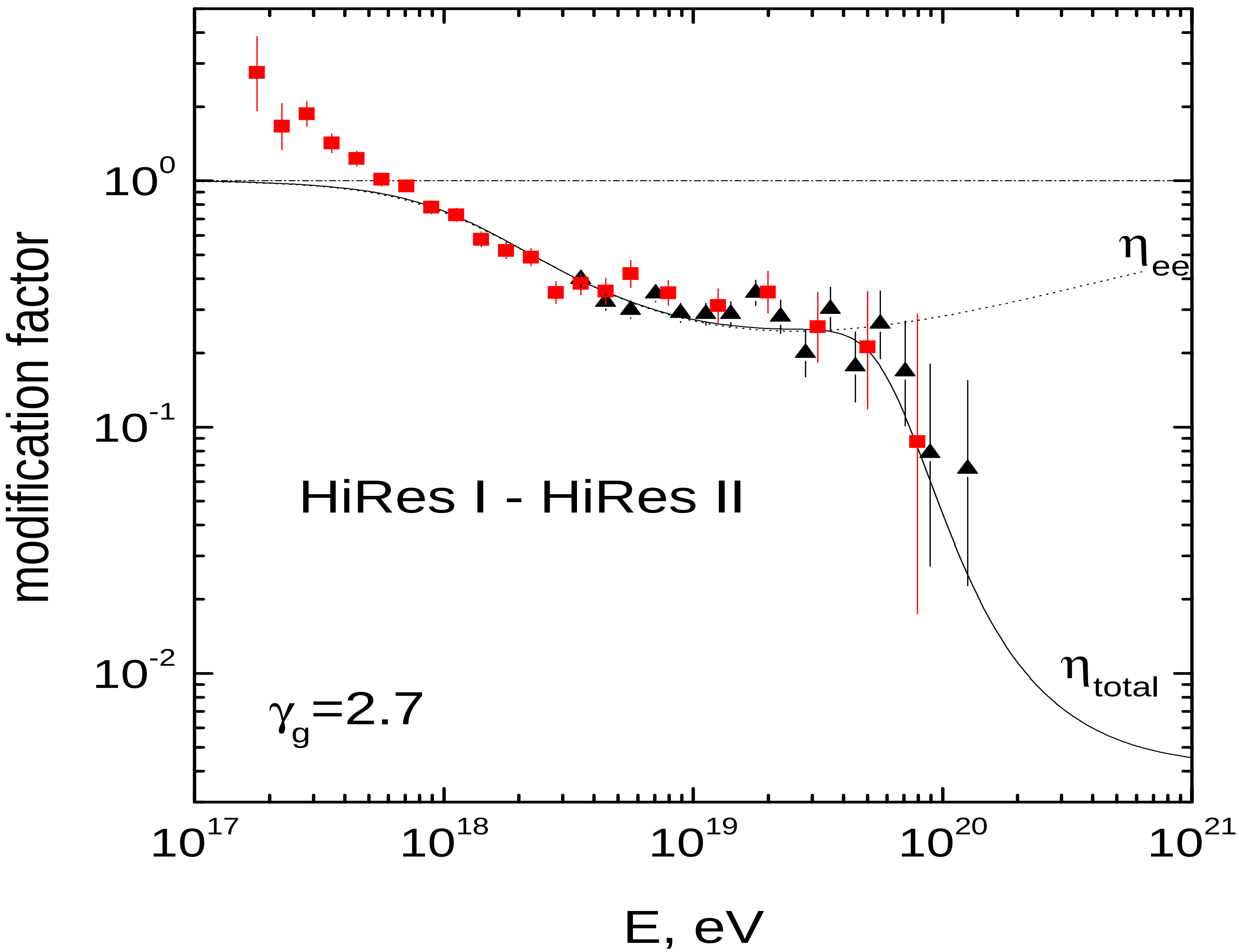
Traditional (from 70s) model of **ankle transition**, $E_a \sim 1 \times 10^{19}$ eV, contradicts to rigidity confinement and acceleration.



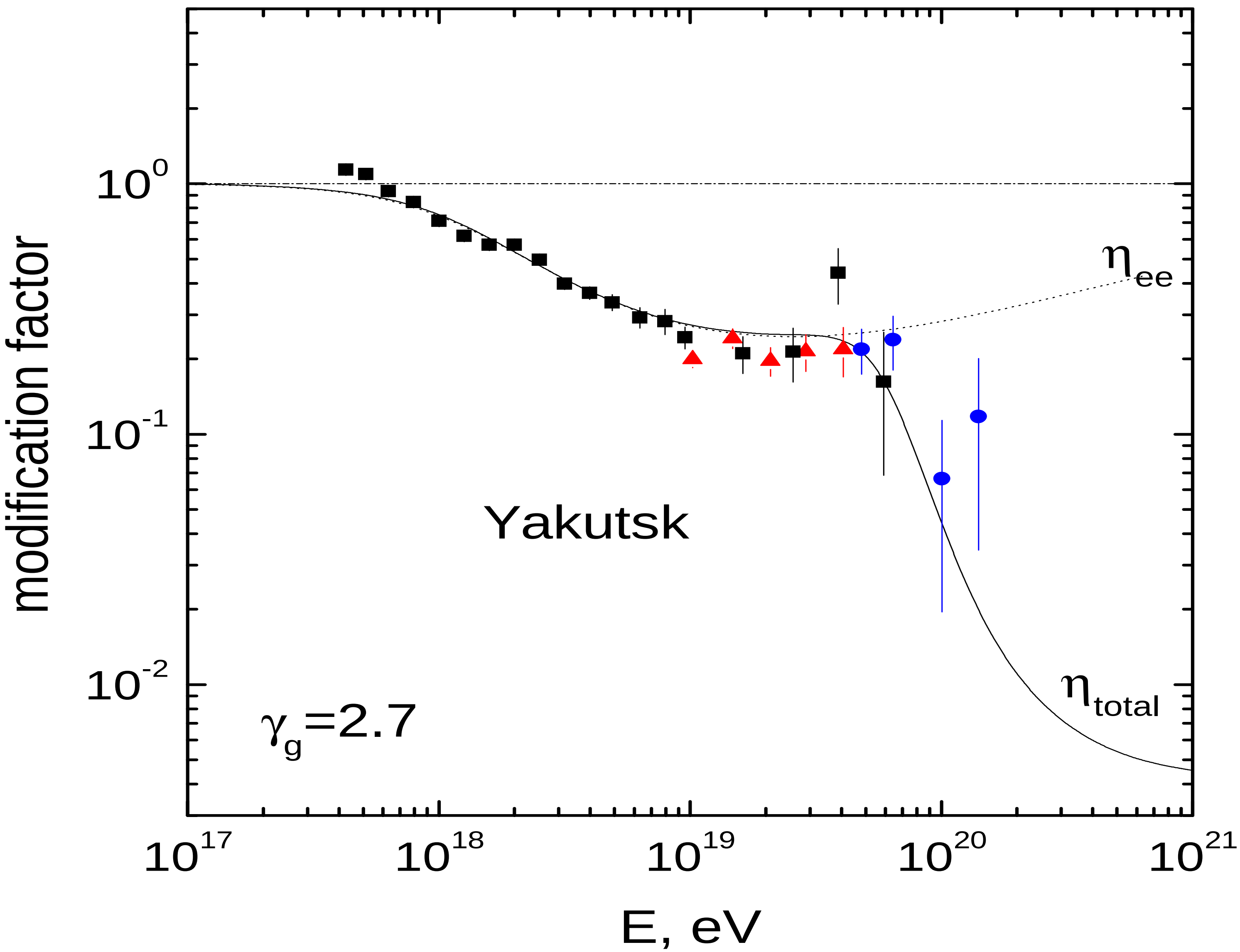
DIP IN COMPARISON WITH AKENO-AGASA DATA



DIP IN COMPARISON WITH HIRES DATA



DIP IN COMPARISON WITH YAKUTSK DATA



DIFFUSION OF UHECR IN EXPANDING UNIVERSE

DIFFUSION EQUATION IN EXPANDING UNIVERSE

Metric: $ds^2 = c^2 dt^2 - a^2(t) d\vec{x}^{\,2} = -g_{\mu\nu} dx^{\mu\nu}$,

$$diag\ g_{\mu\nu} = (-1, a^2, a^2, a^2), \quad diag\ g^{\mu\nu} = (-1, 1/a^2, 1/a^2, 1/a^2),$$

Diffusive flux in the local frame:

$$j_k = -D \frac{\partial}{\partial x^k} n(\vec{x}, t), \quad (k = 1,\ 2,\ 3).$$

Conservation of current j^μ :

$$\frac{\partial}{\partial x^\mu} \left(\sqrt{g} j^\mu \right) = 0.$$

Performing differentiation:

$$\frac{\partial}{\partial t} n(\vec{x}, t) + 3H(t) n(\vec{x}, t) - \frac{D}{a^2} \nabla_x^2 n(\vec{x}, t) = 0,$$

Including energy losses and the source term:

$$\frac{\partial n}{\partial t} + 3H(t) n - \frac{D(E,t)}{a^2(t)} \nabla_x^2 n - \frac{\partial}{\partial E} \left[b(E,t) n \right] = \frac{Q(E,t)}{a^3(t)} \delta^3(\vec{x} - \vec{x}_g).$$

Analytic solution of the diffusion equation

Equation for the Fourier components $f_\omega(E,t)$:

$$\frac{\partial}{\partial t} f_\omega(E,t) - b(E,t)\frac{\partial}{\partial E} f_\omega(E,t) + \left[3H(t) - \frac{\partial b(E,t)}{\partial E} + \vec{\omega}^2 \frac{D(E,t)}{a^2(t)}\right] f_\omega(E,t) = \frac{Q(E,t)}{a^3(t)}.$$

The characteristic equation:

$$dE/dt = -b(E,t)$$

coincides with equation for energy evolution. Its solution is

$$\mathcal{E}' = E'(E,t,t').$$

The solution of equation for $f_\omega(E,t)$ with energies taken on characteristic:

$$f_\omega(E,t) = \int_{t_g}^{t} dt' \frac{Q(\mathcal{E}',t')}{a^3(t')} \exp\left\{ - \int_{t'}^{t} dt'' \left[3H(t'') - \frac{\partial b(\mathcal{E}'',t'')}{\partial \mathcal{E}''} + \vec{\omega}^2 \frac{D(\mathcal{E}'',t'')}{a^2(t'')}\right]\right\}$$

Introducing the analogue of the **Syrovatsky variable**

$$\lambda(E,t') = \int_{t'}^{t} dt'' \frac{D(\mathcal{E}'',t'')}{a^2(t'')},$$

we obtain for spherically symmetric case

$$\mathbf{n(x_g,E)} = \int_0^{\mathbf{z_g}} \mathbf{dz} \left|\frac{\mathbf{dt}}{\mathbf{dz}}\right| \mathbf{Q[E_g(E,z),z]} \frac{\mathbf{exp[-x_g^2/4\lambda(E,z)]}}{\mathbf{[4\pi\lambda(E,z)]^{3/2}}} \frac{\mathbf{dE_g}}{\mathbf{dE}},$$

where

$$\frac{dE_g}{dE} = (1+z)\exp\left[\int_0^z dz' \left|\frac{dt'}{dz'}\right| \frac{\partial b_{int}(\mathcal{E}',z')}{\partial \mathcal{E}'}\right],$$

$$-dt/dz = 1/\left[H_0(1+z)\sqrt{\Omega_m(1+z)^3+\Lambda}\right],$$

to be compared with the Syrovatsky solution:

$$\mathbf{n_S(E,x_g)} = \frac{\mathbf{1}}{\mathbf{b(E)}} \int_{\mathbf{E}}^{\infty} \mathbf{dE_g Q(E_g)} \frac{\mathbf{exp}\left[\mathbf{-x_g^2/4\lambda(E,E_g)}\right]}{\left[\mathbf{4\pi\lambda(E,E_g)}\right]^{\mathbf{3/2}}}.$$

THREE TESTS OF THE SOLUTION

1. The solution coincides with the Syrovatsky solution when

$$D(E,t) = D(E), \;\; b(E,t) = b(E), \;\; a(t) = 1 \; .$$

2. In case of homogeneous distribution of sources, the solution gives the universal spectrum as must be according to propagation theorem.

3. Solution for rectilinear-propagation equation

$$\frac{\partial n}{\partial t} + \frac{c\vec{e}}{a(t)} \frac{\partial n}{\partial \vec{x}} - b(E,t)\frac{\partial n}{\partial E} + 3H(t)n - n\frac{\partial b}{\partial E} = \frac{Q(E,t)}{a^3(t)}\delta^3(\vec{x} - \vec{x}_g),$$

obtained by the same formal method gives the correct (known) solution

$$n(t_0, E) = \frac{Q(E_g, t_g)}{4\pi c x_g^2 (1 + z_g)} \frac{dE_g}{dE}$$

CONCLUSIONS

- **We obtained the analytic solution of diffusion equation for ultra-relativistic ($E \approx p$) particles (electrons, protons, nuclei). The solution is valid for expanding universe and for diffusion coefficient D and energy loss b with arbitrary dependence on E and t.**

- **The method of diffusion equation is important for UHECR at low energies $E \lesssim (1 - 10) \times 10^{18}$ eV, where numerical simulations need unrealistically long computation time.**

- **At $E < 1 \times 10^{18}$ eV spectrum of extragalactic protons has the diffusion cutoff, which provides transition from extragalactic to galactic cosmic rays at the second knee at $E_{\rm 2kn} \sim (0.4 - 0.8) \times 10^{18}$ eV, as measured in different experiments.**

- **The diffusive approximation gives better understanding of UHECR propagation in extragalactic magnetic fields, *e.g.* transition to universal spectrum, absence of GZK cutoff in strong magnetic fields etc.**

  Comparison with numerical simulations (Aloisio and VB 2004 and Yoshiguchi et al 2003) show good agreement in spectra for the range of parameters appropriate for diffusion description.