Application of Picard-Chebyshev method to orbital dynamics

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1 What is Picard-Chebyshev method ?

1.1 Outlines

- Considering the following ordinary differential equation;

$$\frac{dx}{dt} = f(x,t), \quad x(t_0) = x_0$$

- Initially starting from global approximate/rough solution.
- Iteratively converging the solution using Picard iteration method:

$$x^n(t) = x_0 + \int_{t_0}^{t} f(x^{n-1}(t'), t')dt'$$

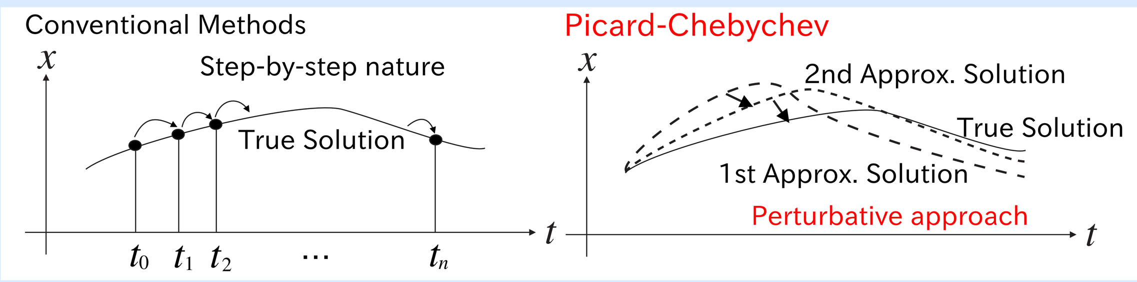
- Generally the integrand is unintegrable then expanding it by Chebyshev polynomials $T_i(t)$s with coefficients F_is,

$$\underbrace{f(x,t)}_{\text{unintegrable}} \Rightarrow \underbrace{F_0T_0(t) + F_1T_1(t) + F_2T_2(t) + \cdots + F_NT_N(t)}_{\text{integrable}}$$

- Chebyshev polynomials are analytically integrable like a trigonometric functions.
- **Therefore the aim of this method is to obtain coefficients X_is and determine the function form of solution in given interval $[a,b]$ as**

$$x(t) = \sum_{i=0}^{N^{(n)}} X_i T_i(t)$$

1.2 Schematic diagram



1.3 Previous work

- Fukushima (1997a) : Application to perturbed harmonic oscillator
- Fukushima (1997b) : Vectorization of perturbed harmonic oscillator
- **In this poster we show Picard-Chebyshev method is useful tool in an orbital dynamics.**

2 Procedure of Picard-Chebyshev method

2.1 Main sequence

1. Letting number of terms of n-th iteration be $N^{(n)}$ and representing as,

$$f^{(n)}(x^{(n)}(t), t) = \sum_{j=0}^{N^{(n)-1}} F_j T_j(\tau), \quad x^{(n)}(t) = \sum_{j=0}^{N^{(n)}} X_j T_j(\tau)$$

2. In this method, function evaluations are done by using zeros of $T_{N^{(n)}}(\tau) = 0$, $\tau_k^{(n)}$;

$$\tau_k^{(n)} = \frac{a+b}{2} - \frac{H}{2}\cos\left(\frac{(2k-1)\pi}{2N^{(n)}}\right), \quad k = 1, \cdots, N^{(n)}, \quad H = b - a$$

and using them,

$$x_k^{(n)} \equiv x^{(n-1)}(\tau^{(n)}{}_k) = \sum_{j=0}^{N^{(n-1)}} c_{jk}^{(n)} X_j^{(n-1)}, \quad c_{jk}^{(n)} = \cos\left(\frac{j(2k-1)\pi}{2N^{(n)}}\right)$$

3. By using $x_k^{(n)}, \tau_k^{(n)}$, evaluating f as $f_k^{(n)} = f(x_k^{(n)}, \tau_k^{(n)})$
4. From orthogonality of discrete Chebyshev Polynomial (Rivlin 1974), the coefficients of f are expressed as,

$$F_0^{(n)} = \frac{1}{N^{(n)}} \sum_{k=1}^{N^{(n)}} f_k, \quad F_j^{(n)} = \frac{2}{N^{(n)}} \sum_{k=1}^{N^{(n)}} c_{jk}^{(n)} f_k, \quad j = 1, \cdots, N^{(n)} - 1$$

5. Using integration formula (Rivlin 1974), coefficients X_is are calculated from F_is. After some improvements to avoid the round-off error, coefficients $X_i^{(n)}$ are given as;

$$X_{N^{(n)}}^{(n)} = \frac{HF_{N^{N^{(n)}-1}}^n}{4N^{(n)}}, \quad X_{N^{(n)}-1}^n = \frac{HF_{N^{N^{(n)}-1}}^n}{4(N^{(n)}-1)}$$

$$X_j^{(n)} = \frac{H}{jN^{(n)}} \sum_{k=1}^{N^{(n)}} s_{jk}^{(n)} g_k^{(n)}, \quad j = 1, \cdots, N^{(n)} - 2$$

$$s_{jk}^{(n)} = \sin\left(\frac{j(2k-1)\pi}{2N^{(n)}}\right), \quad g_k^{(n)} = f_k^{(n)} s_{1k}^{(n)}$$

$$X_0^n = x_0 - \sum_{j=1}^{N^{(n)}} X_j^{(n)} T_j(t'_0)$$

This part is possible to be vectorized/parallelized.

2.2 Determination of $N^{(n)}$

- There is yet empirical way only, and we adopt for n-th iteration,

$$N^{(n)} = \min\left(\frac{nH}{\pi}, N_{\rm opt}\right), \quad N_{\rm opt} = \min M, \quad \sum_{j=M+1}^{N} |X_j| \le \delta$$

δ: accuracy required

2.3 Convergence condition

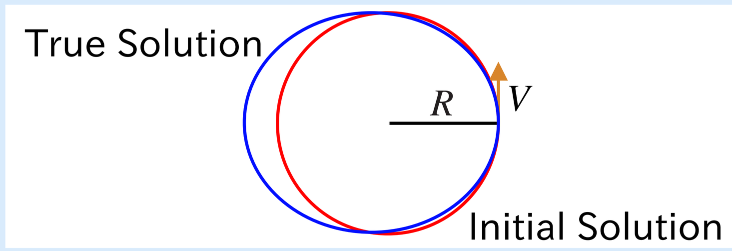
- If $|\Delta x^{(n)}|$ achives the required accuracy, exit the iteration loop;

$$|\Delta x^{(n)}| \equiv |x^{(n)} - x^{(n-1)}| = \sum_{j=0}^{N^{(n-1)}} |X_j^{(n)} - X_j^{(n-1)}| + \sum_{j=N^{(n-1)}+1}^{N^{(n)}} |X_j^n|$$

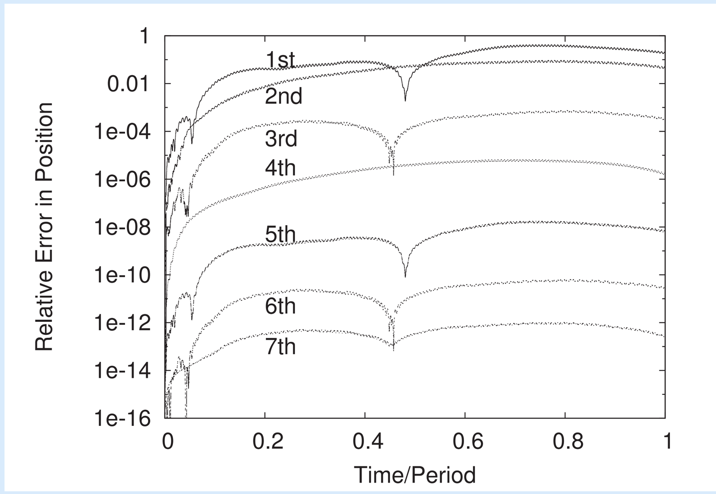
3 Application to Orbital dynamics

3.1 Kepler Problem

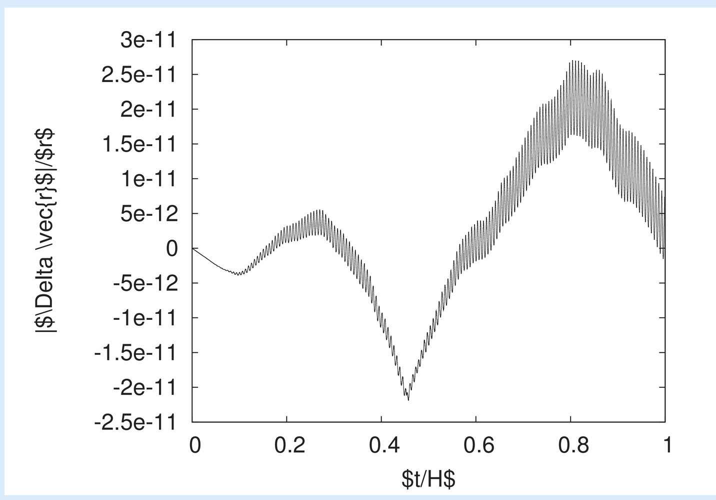
- Adopting the circular solution as a initial (1st approx.) solution,



Positional error in each iteration

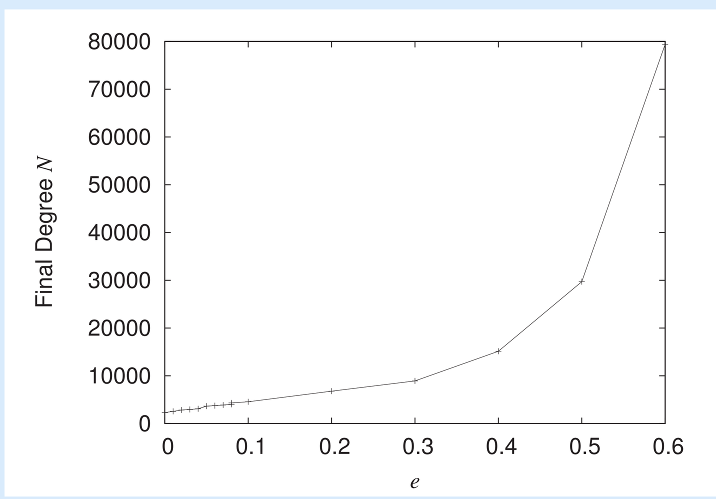


Final positional error in linear scale

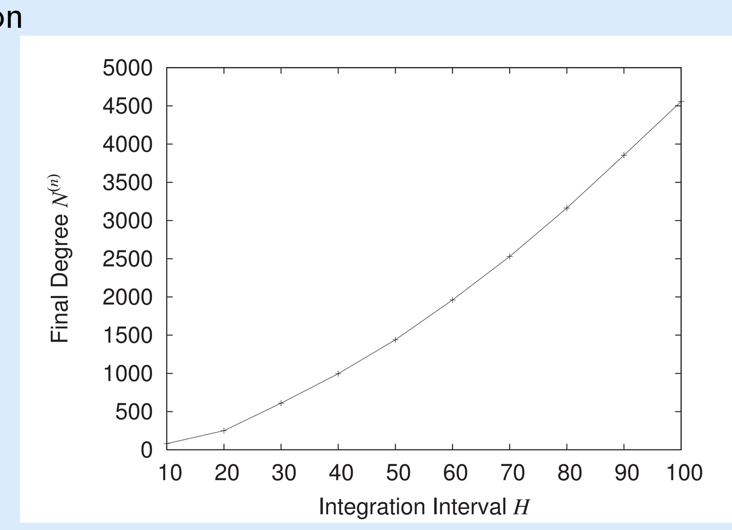


- No secular trend in positional error

e-N relation

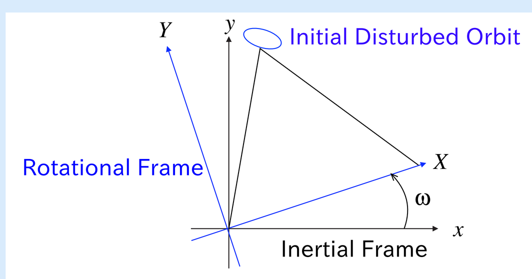


H-N relation

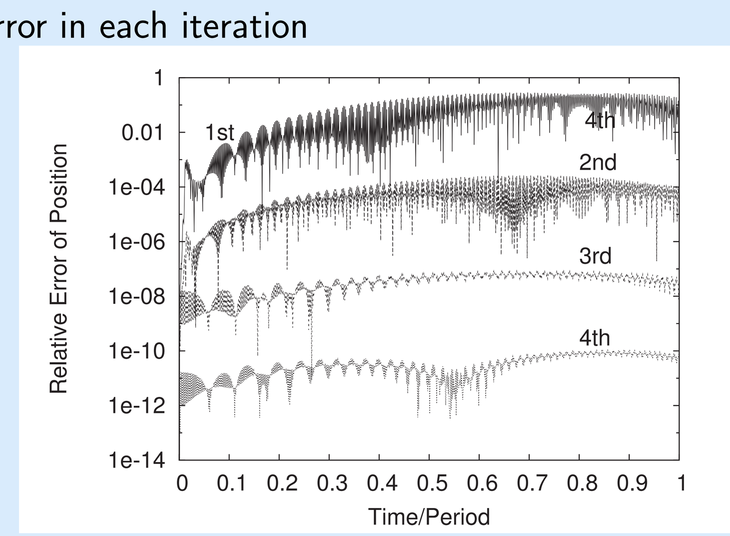


3.2 Three body problem

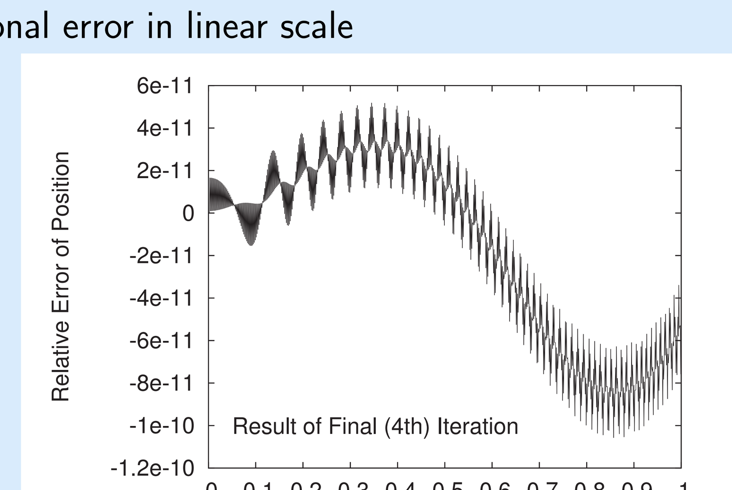
- Motion of planar triangular Lagrange point L_4
- Initially starting from the disturbed orbit



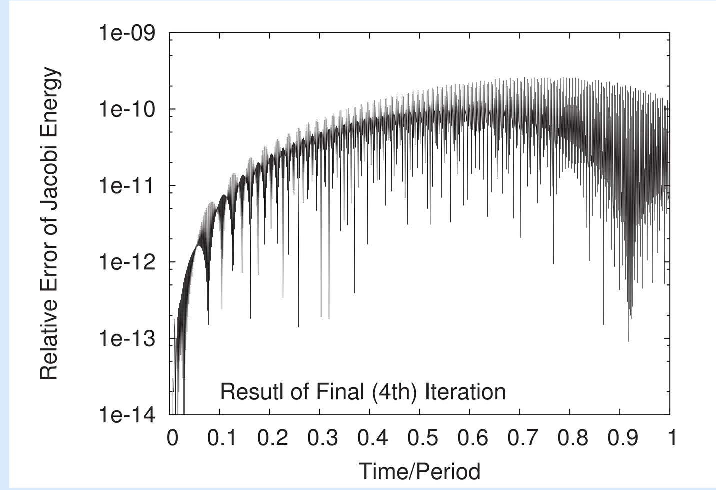
Positional error in each iteration



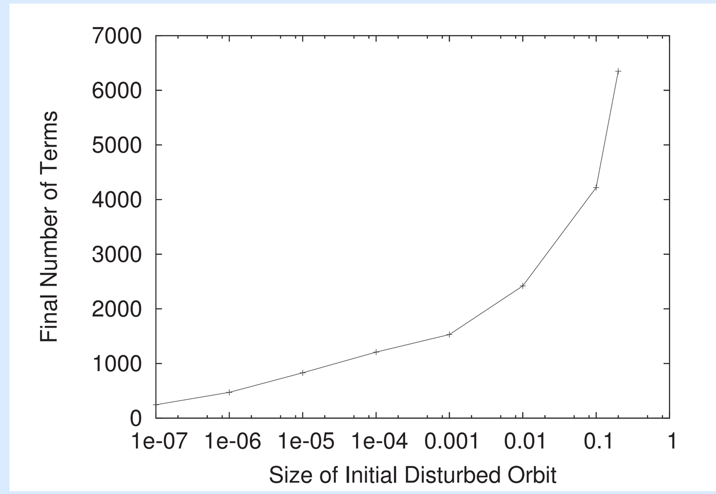
Final positional error in linear scale



Error in conserved quantity (Jacobi constant)



δr-N relation : fixed as $\epsilon = 0.001$



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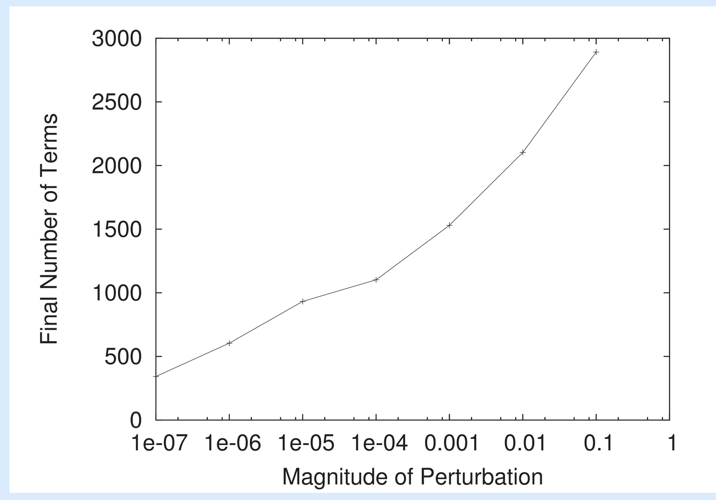
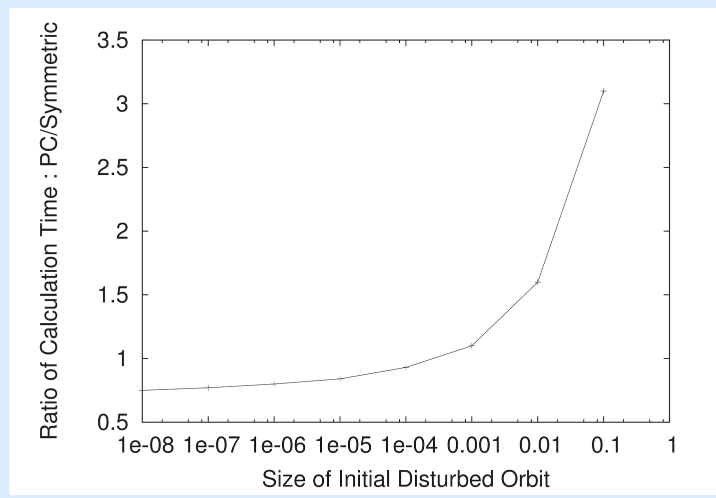


Comparison of calculation time

- Comarison with 8th order symmetric multistep method
- Scalar calculation, $\epsilon = 0.001$



3.3 Vectorization and parallelization

Parallelization of Kepler problem

Number of PE	1	2	4
1/Time	1	1.87	3.65

Vectorization efficiency of Three body problem

$$V = T_{\rm s2}/T_{\rm total} = 0.971, \quad A = T_{\rm s2}/T_{\rm V} = 15.32$$

4 Summary and issues

Summary :

- Picard-Chebychev method works well for orbital dynamics
- In this method, errors in position and conserved quantities show no monotonic trend with respect to time t
  - Usually positional error of numerical integrations increases monotonically with time t as $O(t)$ or $O(t^2)$, and in conserved quantities $O(1)$ or $O(t)$.
  - Chebychev polynomials expressing the solution are quasi-periodic function then residuals also contain the quasi-periodic terms only.
  - Errors in middle range are tend to be larger than in end points. This is because functions are evaluated by using zeros τ_ks which are distributed in end-points more dense than in middle area.
- Possible to speed up by vectorization/parallelization

Issues

- Development of code based on Gauss's planetary equation
- Deriving the determine formula of $N^{(n)}$
- For very long integration, introduction of piecewise Chebychev polynomials and its efficiency
- Completing parallelization of three body problem using MPI

Application

- Long term perturbed dynamics; planets, comets, and satellites
- Verification of general relativistic effects from motion of solar system bodies
- Orbit improvements of ephemeris