MHD Turbulence in Expanding and Contracting Media

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Purpose of Study

1. We investigated MHD turbulence by including the effects of expansion and contraction of background medium.

2. The main goal is to quantify the evolution and saturation of strength and characteristic lengths of magnetic fields in expanding and contracting media.

3. We examine the properties of turbulence in a regime of $t_{eddy} < t_{exp\text{-}cntr}$ and $t_{eddy} > t_{exp\text{-}cntr}$. Based on it, we derive a scaling for the time evolution of flow velocity and magnetic field.

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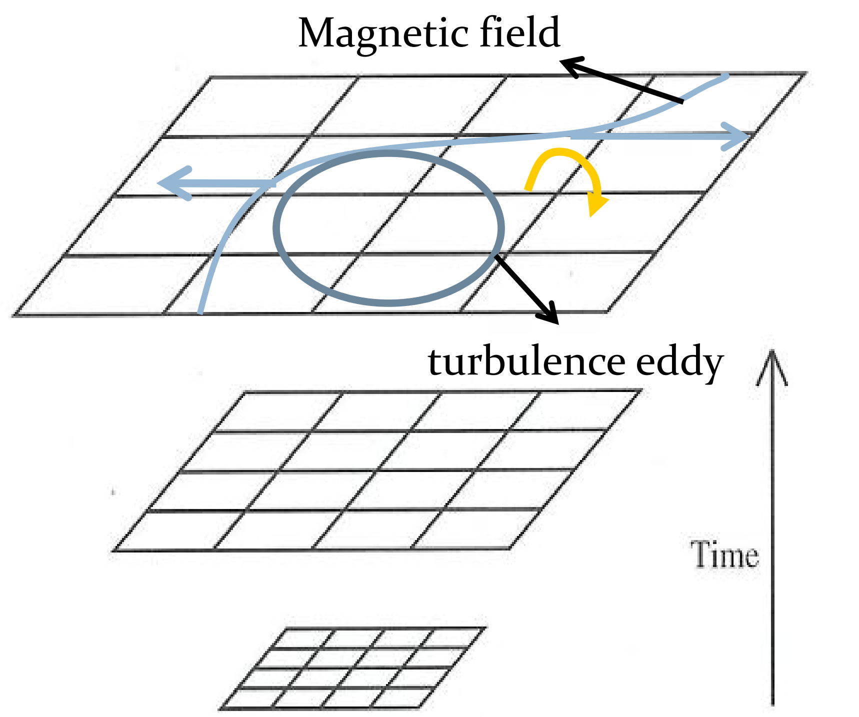
Expansion in the comoving coordinate system



- comoving coordinate system

$$\mathbf{r} = a(t)\mathbf{x}$$

r is physical distance
x is comoving distance
a(t) is scale factor

$$\mathbf{u} = a\dot{\mathbf{x}} + \mathbf{x}\dot{a}$$

$$\mathbf{u} = a\dot{\mathbf{x}} + \mathbf{x}\dot{a}$$

Proper velocity ($\mathbf{x}\dot{a}$)

peculiar velocity ($\mathbf{v} = a\dot{\mathbf{x}}$)

- When the matter expand, the magnetic field in the matter is expand with comoving coordinate system.

The MHD equation in expanding/contracting media

$\mathbf{v}' = \sqrt{\rho}\mathbf{v} = \mathbf{v}/(a\sqrt{a})$ is included density in peculiar velocity and **B** is magnetic field , p' is the pressure , $\mathbf{J} = \nabla \times \mathbf{B}$ is the current , ν is the viscosity, η is the magnetic diffusion. Where f is a random driving force.

$$\frac{\partial \mathbf{v}'}{\partial t} = \sqrt{a}\mathbf{v}' \times (\nabla \times \mathbf{v}') - \frac{5}{2}\frac{\dot{a}}{a}\mathbf{v}' + \sqrt{a}\mathbf{J} \times \mathbf{B} + \frac{1}{a^2}\nu\nabla^2\mathbf{v}' + \nabla P' + f$$

$$\frac{\partial \mathbf{B}}{\partial t} = \sqrt{a}\nabla \times (\mathbf{v}' \times \mathbf{B}) - 2\frac{\dot{a}}{a}\mathbf{B} + \frac{1}{a^2}\nu\nabla^2\mathbf{B}$$

$$a(t) = \left(1 + \frac{t}{t_{\text{exp-cntr}}}\right)^{a_\text{p}}$$

Scale factor with $\text{a}_\text{p} = 1$ for expand, $\text{a}_\text{p} = -1$ for contract

Contracting media

$$a(t) = t_{\text{cntr}}/(t + t_{\text{cntr}})$$
$$\dot{a}(t) = -t_{\text{cntr}}/(t + t_{\text{cntr}})^2$$
$$\dot{a}/a = -1/(t + t_{\text{cntr}})$$

Expanding media

$$a(t) = (t + t_{\text{exp}})/t_{\text{exp}}$$
$$\dot{a}(t) = 1/t_{\text{exp}}$$
$$\dot{a}/a = 1/(t + t_{\text{exp}})$$

When $\text{t}_{\text{exp-cntr}}$ is smaller, the expanding and contracting rate increase.

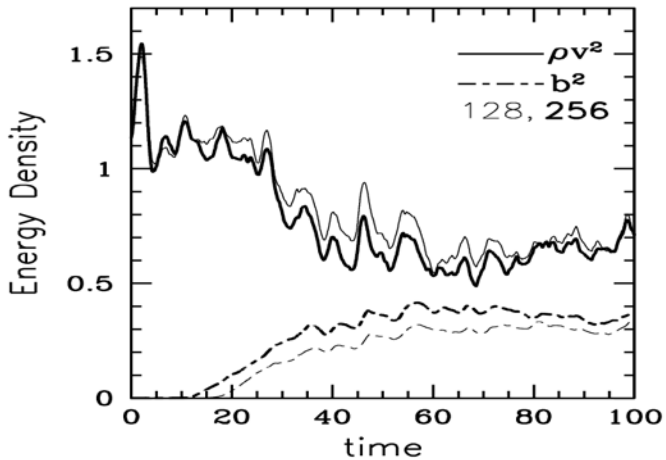
Simulation –initial condition

- Resolution : 128^3 , 256^3 grid (periodic box size = 2π)
- Incompressibility is assumed.
- Have considered only case of $\nu = \mu$
- Have considered hyperviscosity, hyperdiffusion
- At t=0, Mean magnetic field strength is $\mathrm{B}_0 = 0.00001$
- Have simulated using $t_{\mathrm{exp}} = 1, 500$ $t_{\mathrm{cntr}} = 0.1, 10$

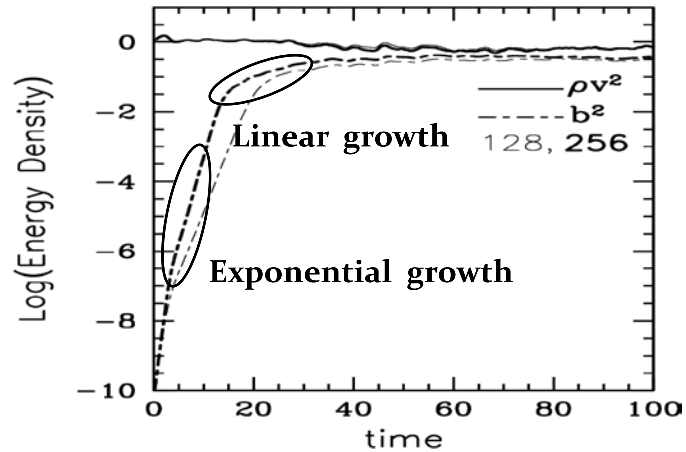
Table 1: Simulation conditions in the case of decaying turbulence

Resolution	Condiction	$t_{\mathrm{exp-cntr}}$	Strength of mean B_0 field
	expand	1, 500	
$128^3, 256^3$	contract	0.1, 10	0.00001
	not expand/contract	∞	

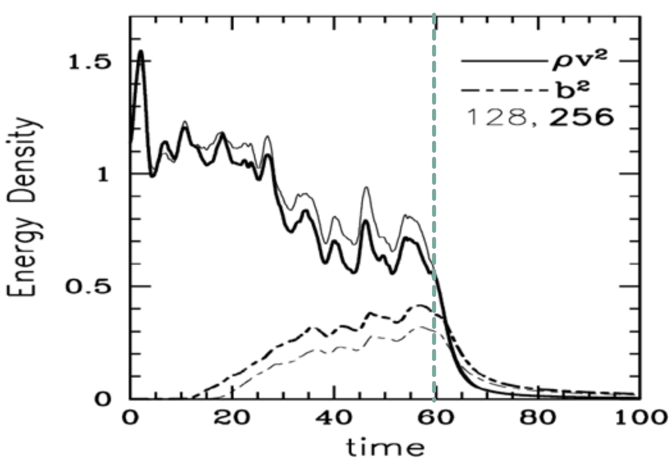
The incompressible MHD turbulence without expanding/contracting effect

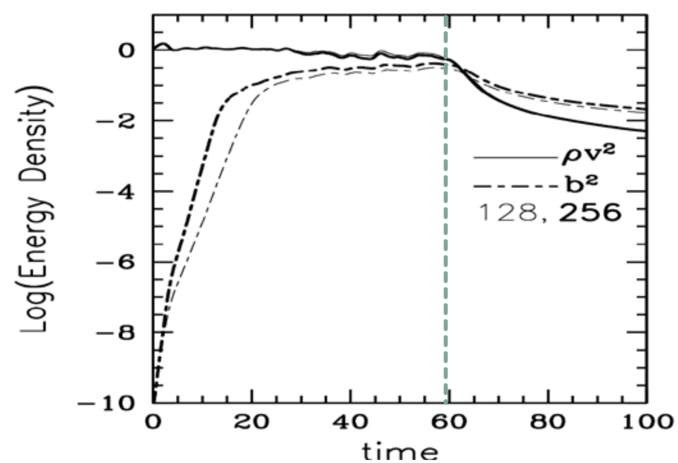


Logarithmic scale



After that the turbulence has reached a stationary state, we turn off the random driving forces.





From this point (t=60) , we let the turbulence decay and inject the effect of the expansion/contraction.

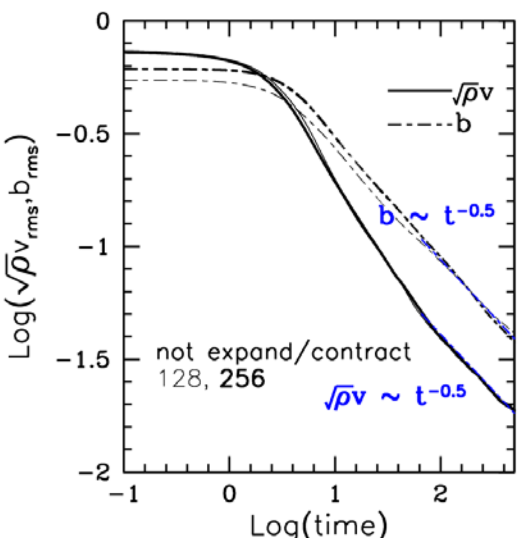
Scaling of velocity and magnetic field

- Three time scales ($t_{eddy} \sim t_{char}$, $t_{exp\text{-}cntr}$)

Decay effect

$$v = v_0\left(1+\frac{t}{t_{char}}\right)^{-\eta}, b = b_0\left(1+\frac{t}{t_{char}}\right)^{-\eta}$$

where t_{char} is the characteristic time of decay ($t_{char} \sim t_{eddy}$)
$\eta = 0.5$ for decaying MHD turbulence ($t \gg t_{char}$)
(Biskamp & Müller 1999)



Expansion/Contraction effects

$$v = v_0/a,\ v' = \sqrt{\rho}v = \frac{v}{a\sqrt{a}} = \frac{v_0}{a^2\sqrt{a}}$$

$$b = b_0/a^2$$

$$\frac{\partial \mathbf{v}'}{\partial t} = \sqrt{a}\mathbf{v}' \times (\nabla \times \mathbf{v}') - \frac{5}{2}\frac{\dot{a}}{a}\mathbf{v}' + \sqrt{a}\mathbf{J} \times \mathbf{B} + \frac{1}{a^2}\nu\nabla^2\mathbf{v}' + \nabla P' + f$$

$$\frac{\partial \mathbf{B}}{\partial t} = \sqrt{a}\nabla \times (\mathbf{v}' \times \mathbf{B}) - 2\frac{\dot{a}}{a}\mathbf{B} + \frac{1}{a^2}\nu\nabla^2\mathbf{B}$$

Scaling of velocity and magnetic field

- Three time scales ($t_{eddy} \sim t_{char}$, $t_{exp\text{-}cntr}$)

Decay effect

$$\mathrm{v} = \mathrm{v}_0 \left(1 + \frac{t}{t_{\mathrm{char}}}\right)^{-\eta}, \mathrm{b} = \mathrm{b}_0 \left(1 + \frac{t}{t_{\mathrm{char}}}\right)^{-\eta}$$

where t_{char} is the characteristic time of decay ($t_{\mathrm{char}} \sim t_{\mathrm{eddy}}$)

$\eta = 0.5$ for decaying MHD turbulence ($t \gg t_{\mathrm{char}}$)

(Biskamp & Müller 1999)

Expansion/Contraction effects

$$\mathrm{v} = \mathrm{v}_0/a, \ \mathrm{v}' = \sqrt{\rho}\mathrm{v} = \frac{\mathrm{v}}{a\sqrt{a}} = \frac{\mathrm{v}_0}{a^2\sqrt{a}}$$

$$\mathrm{b} = \mathrm{b}_0/a^2$$

Decay + Expansion/Contraction effects

$$\mathrm{v}' = \frac{\mathrm{v}_0}{a^2\sqrt{a}} \left(1 + \frac{t}{t_{\mathrm{char}}}\right)^{-\eta}$$

$$\mathrm{b} = \frac{\mathrm{b}_0}{a^2} \left(1 + \frac{t}{t_{\mathrm{char}}}\right)^{-\eta}$$

$$\frac{d\mathrm{v}'}{dt} = -\frac{5}{2}\frac{\dot{a}}{a}\mathrm{v}' - \eta\frac{\mathrm{v}'}{t + t_{\mathrm{char}}}$$

$$\frac{d\mathrm{b}}{dt} = -2\frac{\dot{a}}{a}\mathrm{b} - \eta\frac{\mathrm{b}}{t + t_{\mathrm{char}}}$$

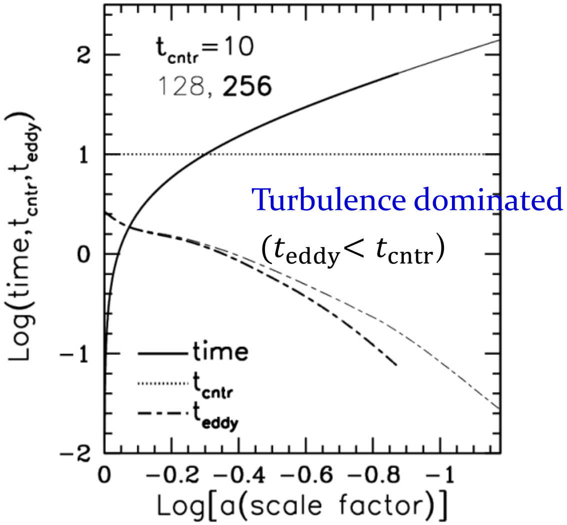
$\dot{a} = da/dt$

$$\frac{d\mathrm{v}'}{da} = -\left(\frac{5}{2} \pm \eta\frac{t + t_{\mathrm{exp-cntr}}}{t + t_{\mathrm{char}}}\right)\frac{\mathrm{v}'}{a}, \qquad \frac{d\mathrm{b}}{da} = -\left(2 \pm \eta\frac{t + t_{\mathrm{exp-cntr}}}{t + t_{\mathrm{char}}}\right)\frac{b}{a}$$

Expanding (+), Contracting (−)

▣ Contracting media

$$\frac{dv'}{da} = -\left(\frac{5}{2} - \eta \frac{t + t_{\text{cntr}}}{t + t_{\text{char}}}\right)\frac{v'}{a}, \qquad \frac{db}{da} = -\left(2 - \eta \frac{t + t_{\text{cntr}}}{t + t_{\text{char}}}\right)\frac{b}{a}$$

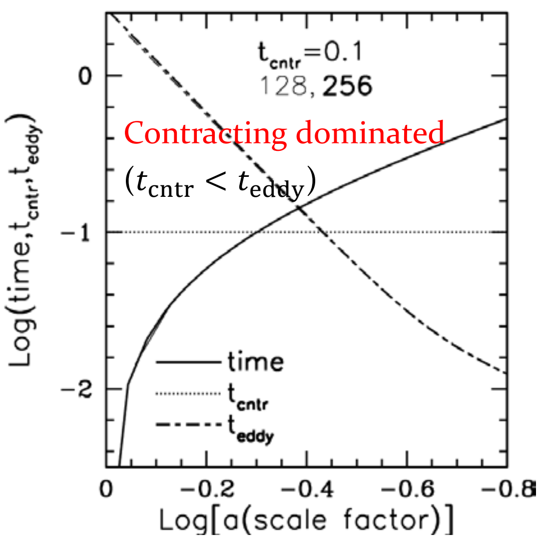


$$a = \left(1 + \frac{t}{t_{\text{cntr}}}\right)^{-1} \propto 1/t$$

- t_{eddy} decreases with the scale factor evolution.

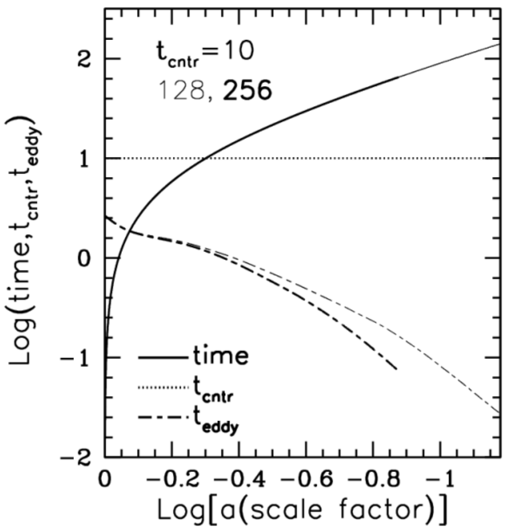
  $$t_{\text{eddy}} = aL_0/(\rho v^2 + b^2)^{1/2}$$

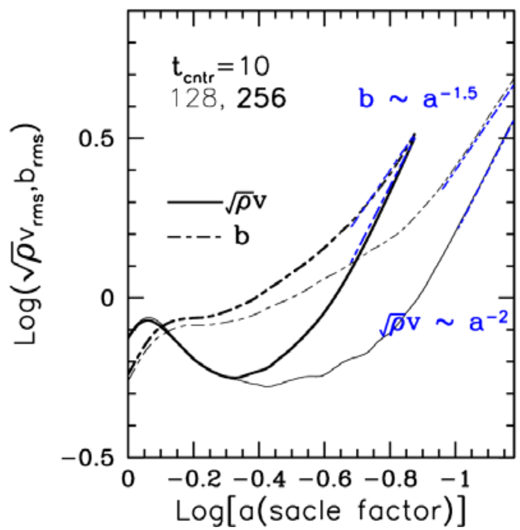
- t_{cntr} is smaller, the contracting rate increase.



▣ Contracting media

$$\frac{dv'}{da} = -\left(\frac{5}{2} - \eta\frac{t + t_{\rm cntr}}{t + t_{\rm char}}\right)\frac{v'}{a}, \qquad \frac{db}{da} = -\left(2 - \eta\frac{t + t_{\rm cntr}}{t + t_{\rm char}}\right)\frac{b}{a}$$



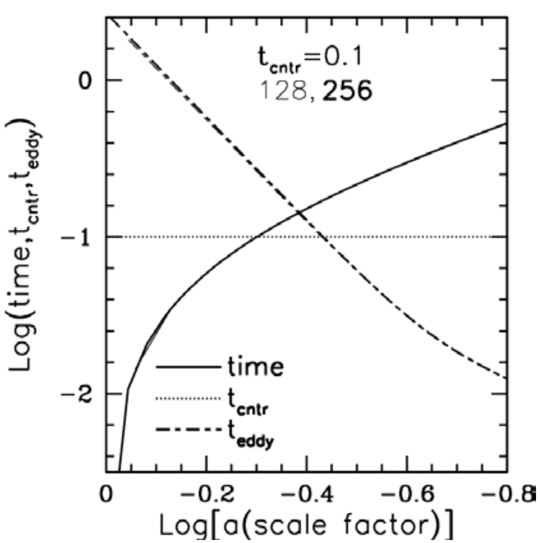


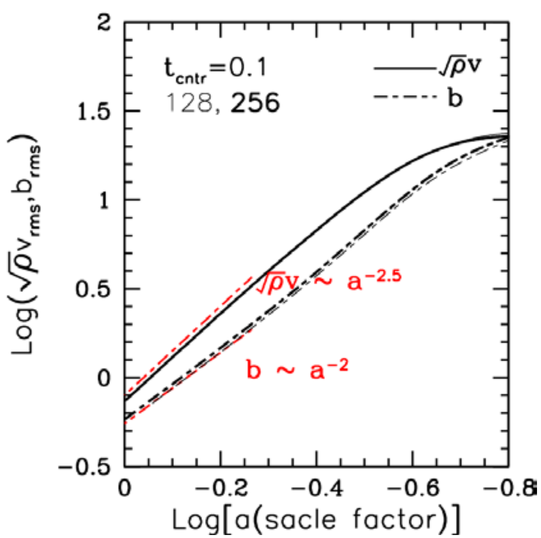
Turbulence dominated

$(t_{\rm eddy} \sim t_{\rm char} < t_{\rm cntr} < t)$

$$\Rightarrow \frac{dv'}{da} = -\left(\frac{5}{2} - \eta\right)\frac{v'}{a}, \frac{db}{da} = -(2-\eta)\frac{b}{a}$$

where $\eta = 0.5$ for decaying MHD turbulence (Biskamp & Müller 1999)





Contracting dominated

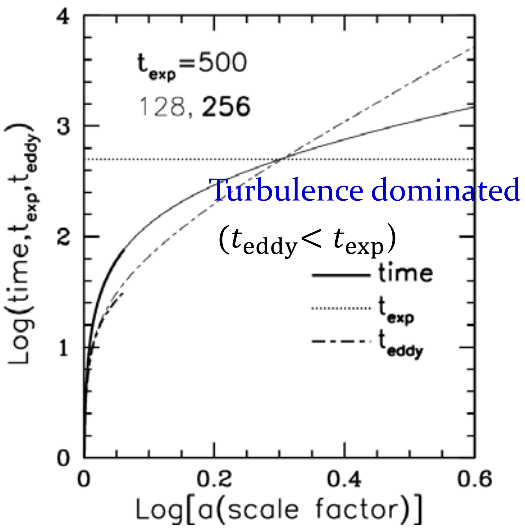
$(t < t_{\rm cntr} < t_{\rm eddy} \sim t_{\rm char})$

$$\Rightarrow \frac{dv'}{da} = -\frac{5}{2}\frac{v'}{a}, \qquad \frac{db}{da} = -2\frac{b}{a}$$

(Robertson & Goldreich 2012)

▣ Expanding media

$$\frac{dv'}{da} = -\left(\frac{5}{2} + \eta \frac{t + t_{\text{exp}}}{t + t_{\text{char}}}\right)\frac{v'}{a}, \qquad \frac{db}{da} = -\left(2 + \eta \frac{t + t_{\text{exp}}}{t + t_{\text{char}}}\right)\frac{b}{a}$$

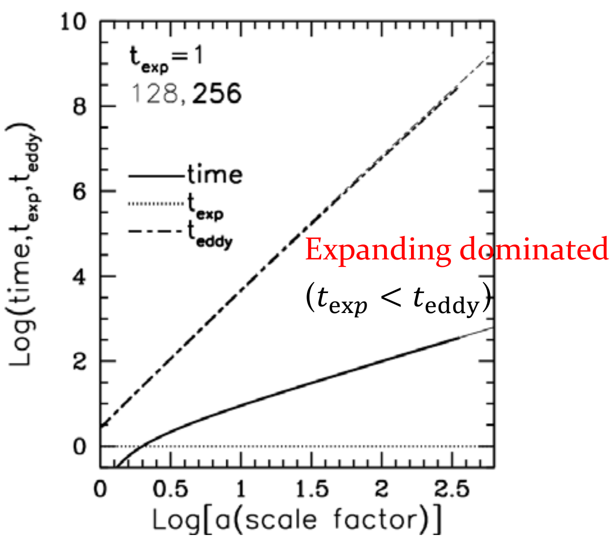


$$a = \left(1 + \frac{t}{t_{\text{exp}}}\right)^1 \propto t$$

- t_{eddy} increases with the scale factor evolution.

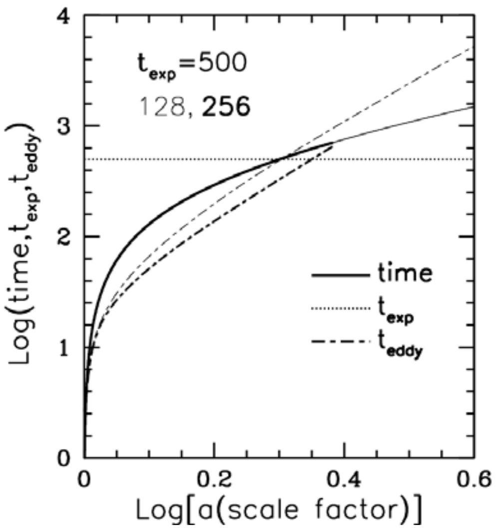
  $$t_{\text{eddy}} = a L_0/(\rho v^2 + b^2)^{1/2}$$

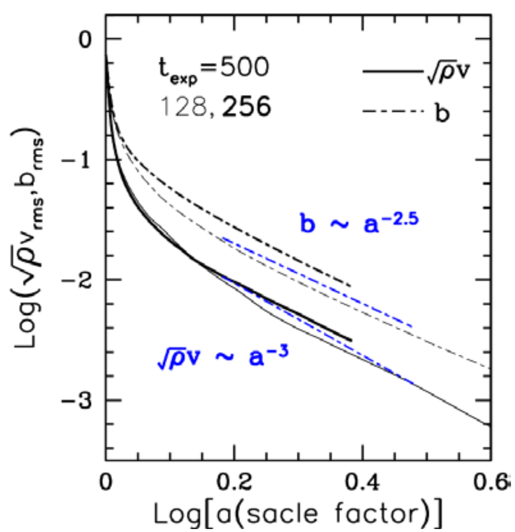
- t_{exp} is smaller, the expanding rate increase.



▣ Expanding media

$$\frac{dv'}{da} = -\left(\frac{5}{2} + \eta\frac{t + t_{\rm exp}}{t + t_{\rm char}}\right)\frac{v'}{a}, \qquad \frac{db}{da} = -\left(2 + \eta\frac{t + t_{\rm exp}}{t + t_{\rm char}}\right)\frac{b}{a}$$



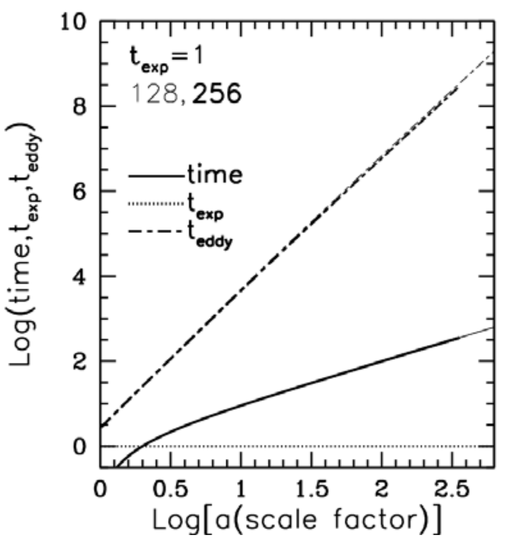


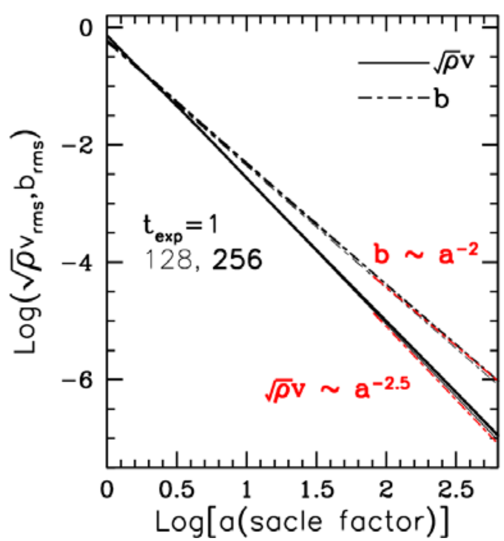
Turbulence dominated

$(t_{\rm eddy} \sim t_{\rm char} \sim t \sim t_{\rm exp})$

⇨ $\frac{dv'}{da} = -\left(\frac{5}{2} + \eta\right)\frac{v'}{a}, \frac{db}{da} = -(2 + \eta)\frac{b}{a}$

where $\eta = 0.5$ for decaying MHD turbulence (Biskamp & Müller 1999)





Expanding dominated

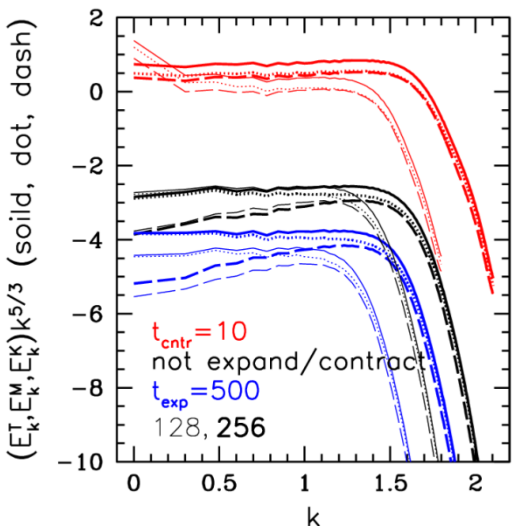
$(t_{\rm exp} < t < t_{\rm eddy} \sim t_{\rm char})$

⇨ $\frac{dv'}{da} = -\frac{5}{2}\frac{v'}{a}, \qquad \frac{db}{da} = -2\frac{b}{a}$

(Robertson & Goldreich 2012)

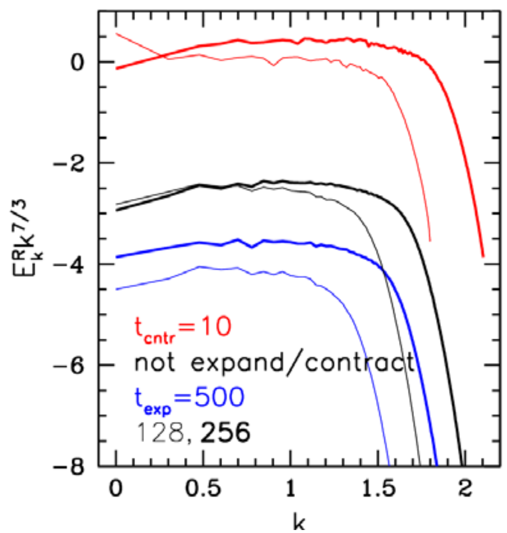
Energy spectrum (Turbulence dominated)

▣ Total energy spectrum



$$E_k^T = E_k^{\mathrm{K}} + E_k^{\mathrm{M}}$$

▣ Residual energy spectrum



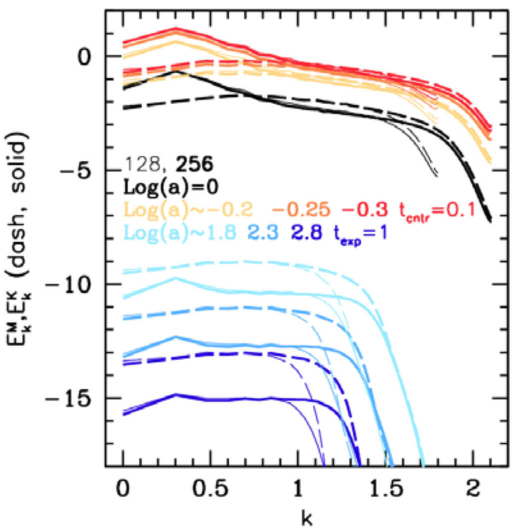
$$E_k^R = E_k^{\mathrm{M}} - E_k^{\mathrm{K}}$$

- Spectrum was obtained by averaging with decaying dominant period. Left and right panels were compensated by $\mathrm{k}^{5/3}$ and $\mathrm{k}^{7/3}$, respectively.

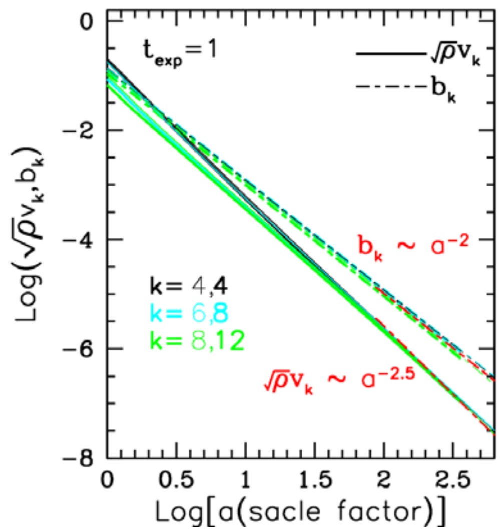
- Regardless of the expansion and contraction effects, the total and residual energy spectrum follow the $E_k^{\mathrm{T}} \sim k^{-5/3}$ and $E_k^{\mathrm{R}} \sim k^{-7/3}$ (Müller 2005) in the inertial range.

Energy spectrum (Expanding/Contracting dominated)

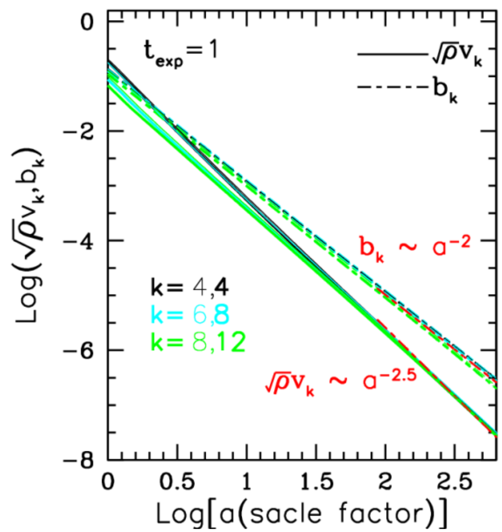
▣ Energy spectrum



▣ Expanding media



▣ Contracting media



$$\frac{dv'_k}{da} = -\frac{5}{2}\frac{v'_k}{a}, \frac{db_k}{da} = -2\frac{b_k}{a} \Rightarrow v'_k \sim a^{-2.5}, b_k \sim a^{-2}$$

- The dissipation range moves toward (larger wave number in contracting media / smaller wave number in expanding media
- Energy spectrum follows the $E_k^K \sim a^{-5}$, $E_k^M \sim a^{-4}$. Because the scale factor dose not depend on wavenumber

Conclusion

- We performed a preliminary study of incompressible MHD decaying turbulence by including the effect of expansion and contraction.

- Scaling of velocity and magnetic field and Spectrum in expanding/contracting media
  - expansion /contraction dominated

    ⇨ $\sqrt{\rho}\text{v}\sim a^{-2.5}, \text{b}\sim a^{-2}$ (Robertson & Goldreich 2012)

    ⇨ the kinetic and magnetic energy spectra follow the $E_k^{\text{K}}\sim a^{-5}$ and $E_k^{\text{M}}\sim a^{-4}$

  - turbulence dominated

    ⇨ $\sqrt{\rho}\text{v}\sim a^{-2.5-\eta}, \text{b}\sim a^{-2-\eta}$ in expanding media
    $\sqrt{\rho}\text{v}\sim a^{-2.5+\eta}, \text{b}\sim a^{-2+\eta}$ in contracting media
    where $\eta = 0.5$ for decaying MHD turbulence (Biskamp & Müller 1999)

    ⇨ the total and residual energy spectra follow the $E_k^{\text{T}}\sim k^{-5/3}$ and $E_k^{\text{R}}\sim k^{-7/3}$ (Müller 2005) in the inertial range.

- The specific results would depend on t_{eddy} and $t_{exp\text{-}cntr}$. We will explore those in future.

The MHD equations in expanding/contracting media

$$\left(\frac{\partial \mathbf{u}}{\partial t}\right)_{\mathbf{r}} + (\mathbf{u}\cdot\nabla_{\mathbf{r}})\mathbf{u} = -\frac{1}{\rho}\nabla_{\mathbf{r}} p - \nabla_{\mathbf{r}}\Phi + \frac{1}{\rho}\mathbf{J}\times\mathbf{B} + \nu\,\nabla_{\mathbf{r}}^2\mathbf{u}$$

$$\mathrm{u} = \dot{a}\mathrm{x} + \mathrm{v}(\mathrm{x},t) = (\dot{a}/a)\mathrm{r} + \frac{1}{\sqrt{\rho}}\mathrm{v}'(\mathrm{r}/a,t)$$ On changing variables from $\mathbf{r}$ to $\mathbf{x} = \mathbf{r}/a$

$$\nabla_r \Rightarrow \frac{1}{a}\nabla \quad , \quad \phi = \Phi + \frac{1}{2}a\ddot{a}x^2$$

Left-hand side
$$\ddot{a}\mathrm{x} + \frac{1}{\sqrt{\rho}}\frac{\partial \mathrm{v}'}{\partial t} + \frac{1}{\sqrt{\rho}}\frac{\dot{a}}{a}\mathrm{v}' + \frac{1}{a\rho}(\mathrm{v}'\cdot\nabla)\mathrm{v}' + \frac{3}{2}\frac{\mathrm{v}'}{\sqrt{\rho}}\frac{\dot{a}}{a}$$

Right-hand side
$$\frac{1}{\rho a}\nabla P - \frac{1}{a}\nabla\phi + \ddot{a}\mathrm{x} + \frac{1}{\rho a}\mathrm{J}\times\mathrm{B} + \frac{1}{a^2\sqrt{\rho}}\nu\,\nabla^2\mathrm{v}'$$

MHD equations in expanding/contracting coordinate

$$\frac{\partial \mathbf{v}'}{\partial t} = \sqrt{a}\mathbf{v}'\times(\nabla\times\mathbf{v}') - \frac{5}{2}\frac{\dot{a}}{a}\mathbf{v}' + \sqrt{a}\mathbf{J}\times\mathbf{B} + \frac{1}{a^2}\nu\nabla^2\mathbf{v}' + \nabla p' + f$$

The MHD equations in expanding/contracting media

$$\left(\frac{\partial \mathbf{B}}{\partial t}\right)_{\mathbf{r}} = \nabla_{\mathbf{r}} \times (\mathbf{u} \times \mathbf{B}) + \eta\, \nabla_{\mathbf{r}}^2 \mathbf{B}$$

$$\mathrm{u} = \dot{a}\mathrm{x} + \mathrm{v}(\mathrm{x},t) = (\dot{a}/a)\mathrm{r} + \frac{1}{\sqrt{\rho}}\mathrm{v}'(\mathrm{r}/a,t)$$ On changing variables from $\mathbf{r}$ to $\mathbf{x} = \mathbf{r}/a$

$$\nabla_r \Rightarrow \frac{1}{a}\nabla$$

Rule of Multiplication $\nabla \times (A \times B) = (B\cdot\nabla)A - (A\cdot\nabla)B + A(\nabla\cdot B) - B(\nabla\cdot A)$

$$\frac{\partial \mathrm{B}}{\partial t} = (\mathrm{B}\cdot\nabla_{\mathrm{r}})\mathrm{u} - (\mathrm{u}\cdot\nabla_{\mathrm{r}})\mathrm{B} + \mathrm{u}(\nabla_{\mathrm{r}}\cdot\mathrm{B}) - \mathrm{B}(\nabla_{\mathrm{r}}\cdot\mathrm{u}) + \eta\, \nabla_{\mathrm{r}}^2\mathrm{B}$$

$$= \frac{1}{a\sqrt{\rho}}(\mathrm{B}\cdot\nabla)\mathrm{v}' - \frac{\dot{a}}{a}\mathrm{B} - \frac{1}{a\sqrt{\rho}}(\mathrm{v}'\cdot\nabla)\mathrm{B} + \frac{1}{a\sqrt{\rho}}\mathrm{v}'(\nabla\cdot\mathrm{B}) - \mathrm{B}\frac{\dot{a}}{a} - \frac{1}{a\sqrt{\rho}}(\nabla\cdot\mathrm{v}')\mathrm{B} + \frac{1}{a^2}\eta\, \nabla^2\mathrm{B}$$

Magnetic field equations in expanding/contracting coordinate

$$\frac{\partial \mathrm{B}}{\partial t} = \sqrt{a}\nabla \times (\mathrm{v}'\times\mathrm{B}) - 2\frac{\dot{a}}{a}\mathrm{B} + \frac{1}{a^2}\eta\nabla^2\mathrm{B}$$