Inverse Cascade in Imbalanced Electron Magnetohydrodynamic (EMHD) Turbulence

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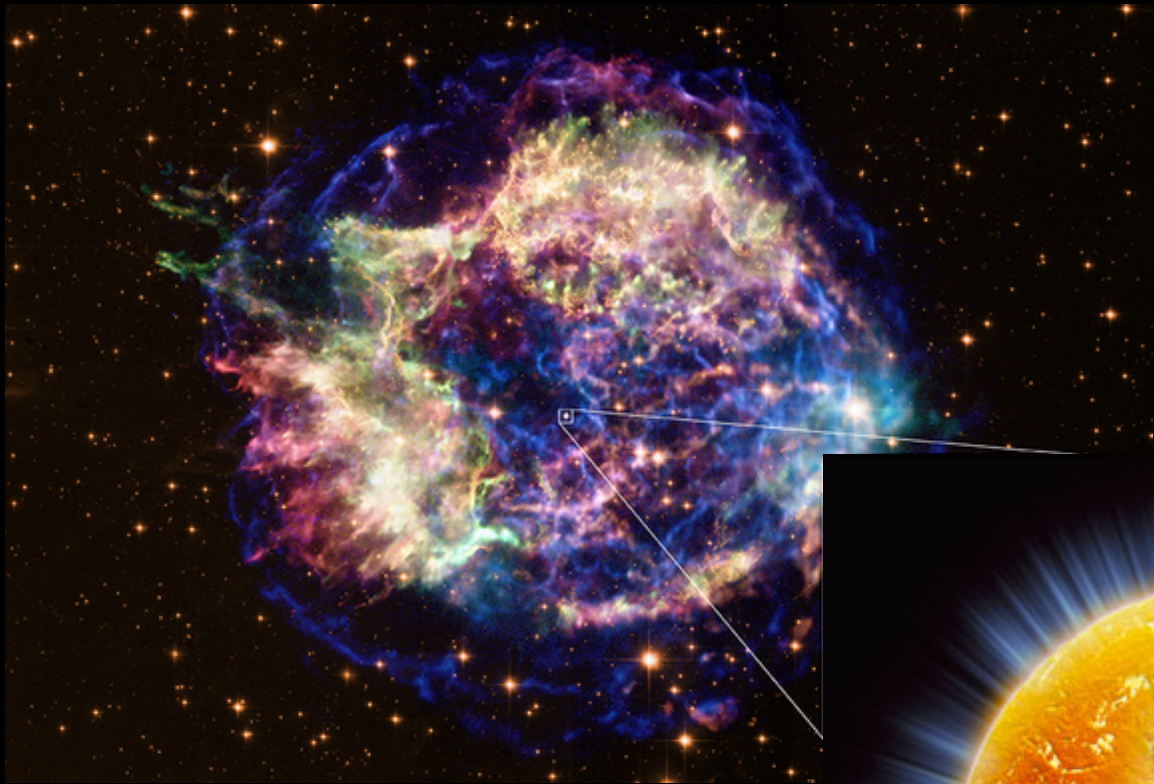
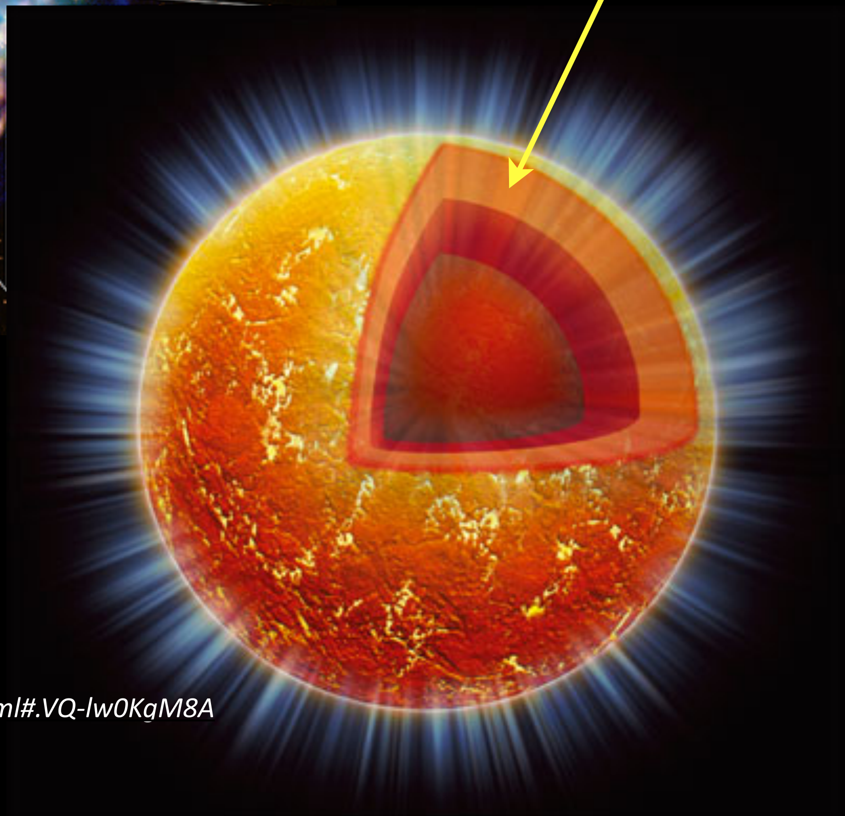
Introduction





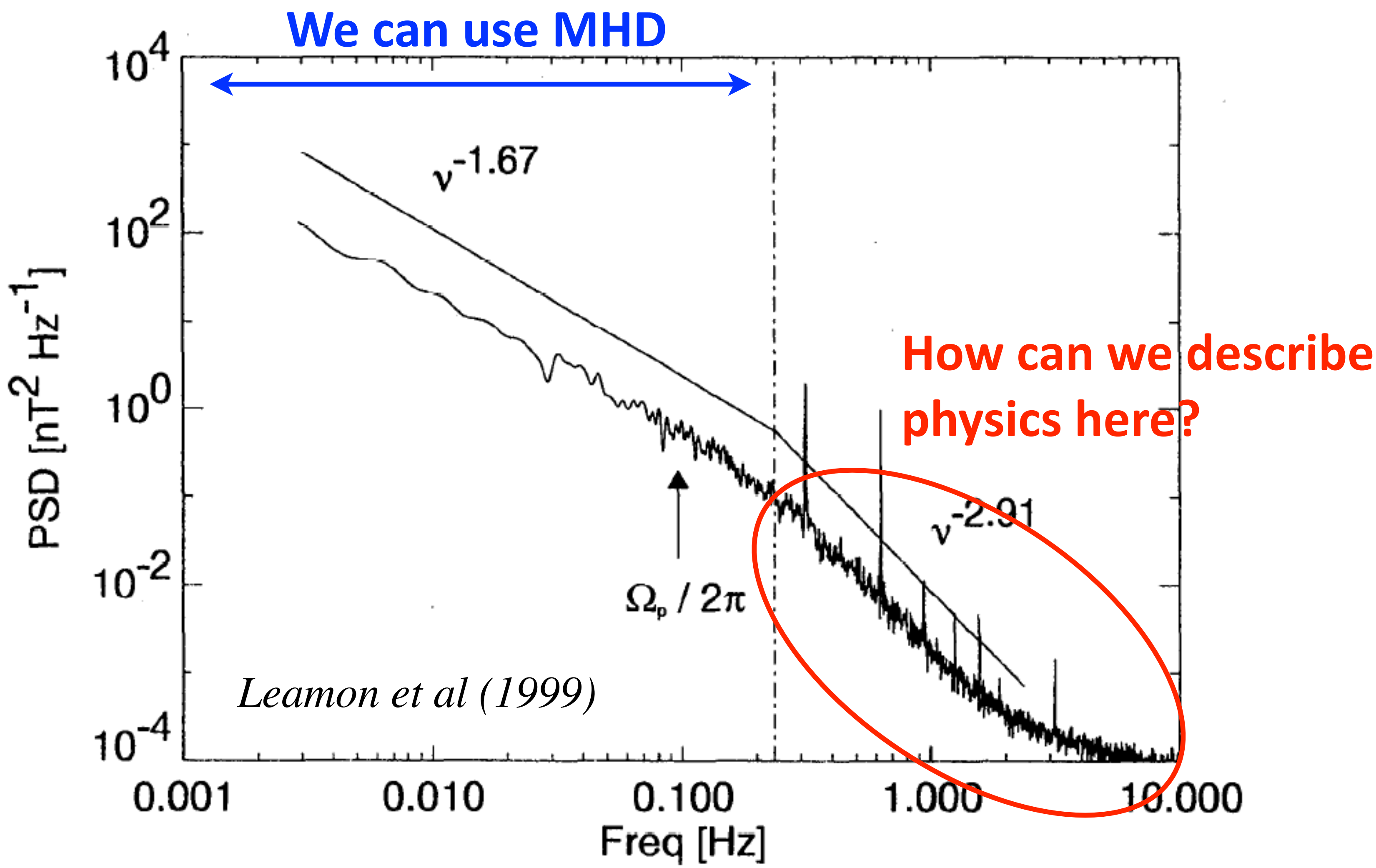
Neutron star crust

How can we describe physics here?

https://www.nasa.gov/mission_pages/chandra/multimedia/casa2011.html#.VQ-lw0KgM8A

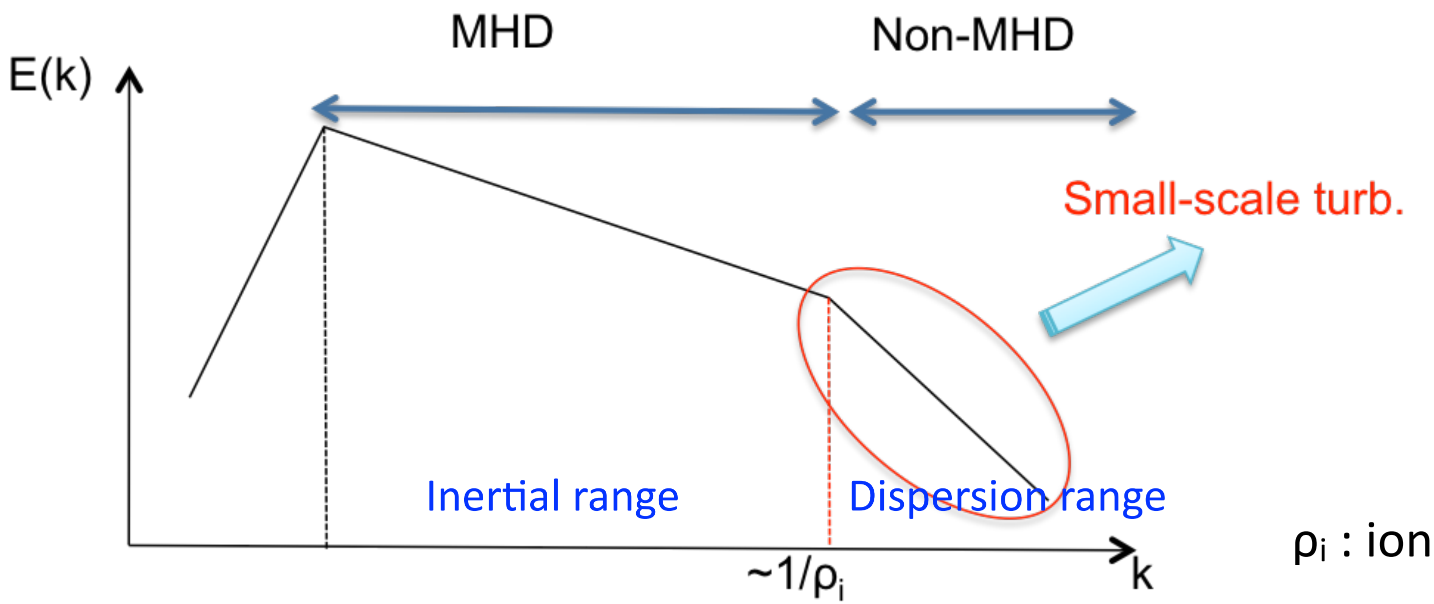
Introduction

Solar wind power spectrum



Outline

- On large scale (e.g. plasmas as conducting fluids)
  → use MHD
  → not suitable for describing **small-scale** (near and below proton **gyro-scale**)

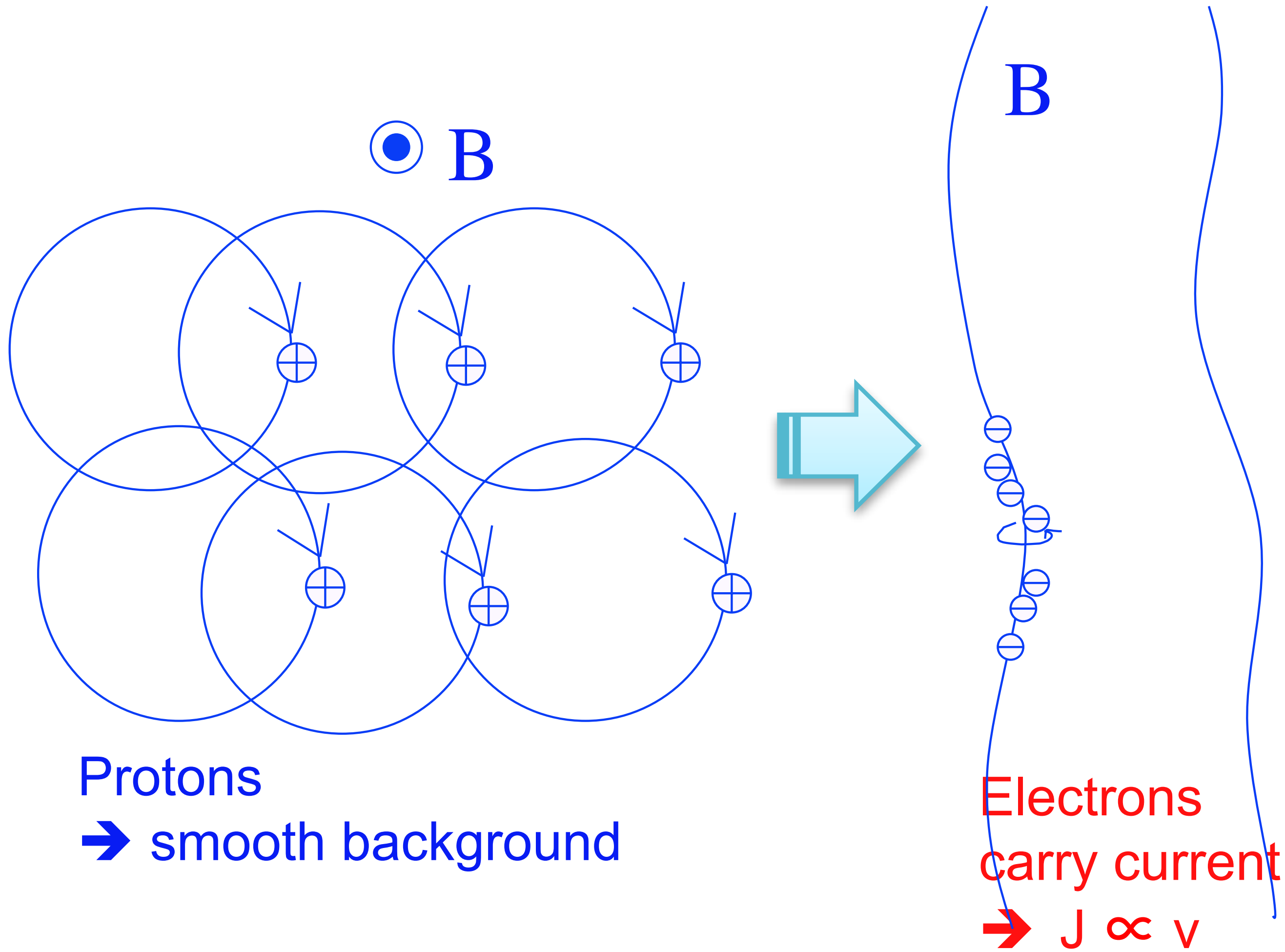


ρ_i : ion gyro-scale

- **How can we deal with small-scale physics?**
  → EMHD
  → a fluid-like model of small-scale plasmas
  (cf. PIC or gyro-kinetic simulations)
  → Hall MHD in the limit of infinite gyro-radius (*A. S. Kingsep, et. Al., 1990*).

EMHD Introduction

- A simple fluid-like description of small-scale physics.
- **Electron MHD equation** ← magnetic induction equation.



EMHD equation

$$\mathbf{J} \propto \mathbf{v}$$

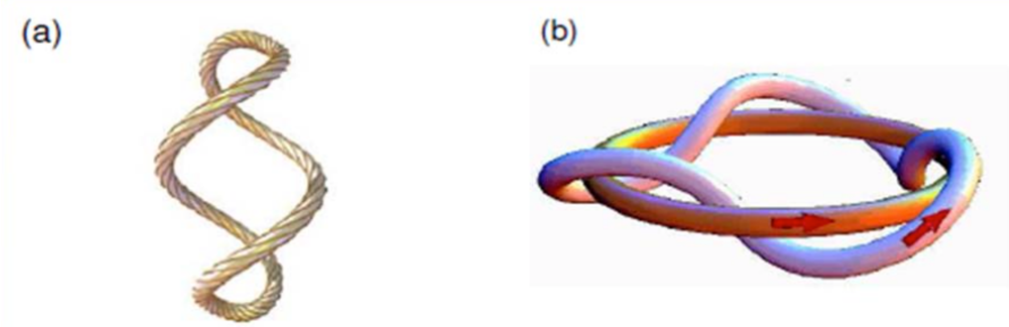
$$\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\cancelto{0}{\frac{\partial \mathbf{E}}{\partial t}}$$

$$\boxed{\mathbf{v} \propto \nabla \times \mathbf{B}}$$

inserting
into induction eq.

$$\boxed{\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times [(\nabla \times \mathbf{B}) \times \mathbf{B}] + \eta' \nabla^2 \mathbf{B}}$$

Definition of Magnetic helicity



- Magnetic helicity is **a measure of** how much magnetic field lines in a flux tube are twisted around the tube axis (***twist***), how much the tube axis in kinked (***kinks***), and how much flux tubes are interlinked with each other in a magnetic field system (***inter-linkages***).

- **Mathematically**, magnetic helicity *H*, inside a volume *V*, of a magnetic field system is defined by:

$$\boxed{H = \int_V \mathbf{A} \bullet \mathbf{B}\, dV,}$$

where **A** is the vector potential of magnetic field **B**

i.e. $\mathbf{B} = \nabla \times \mathbf{A}$

Conserved quantities in MHD & EMHD

- If the magnetic diffusivity is zero, both magnetic energy and magnetic helicity are conserved quantities.

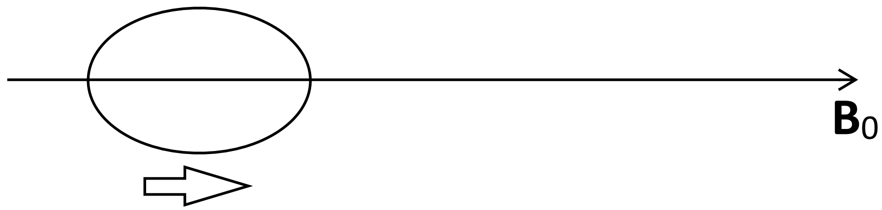
- If the magnetic diffusivity is not zero,

  - It is possible to destroy magnetic energy through turbulence cascade

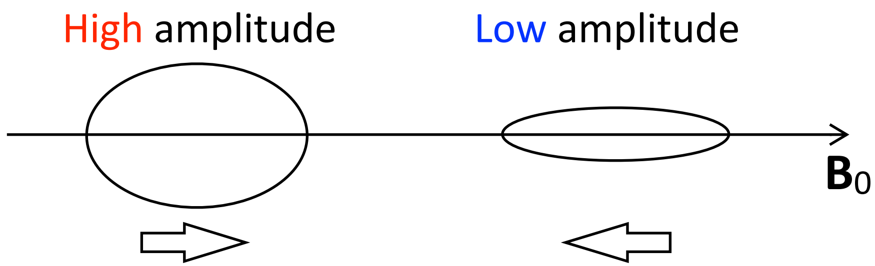
  - Since **magnetic helicity** is a better **conserved quantity** than magnetic energy, it is difficult to destroy magnetic helicity even in the presence of turbulence.

What is Imbalanced Turbulence?

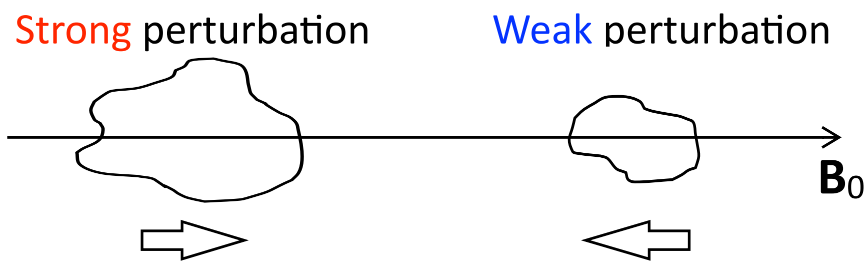
- In MHD, wave packets propagate along B lines & become elongate along B.



- Imbalanced turbulence : the energies of waves traveling in opposite directions along the magnetic field line are not equal.

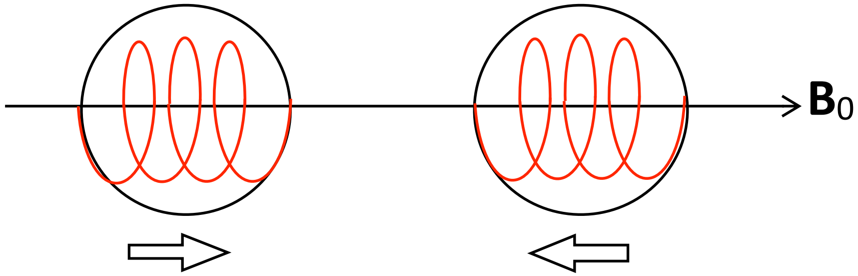


e.g)

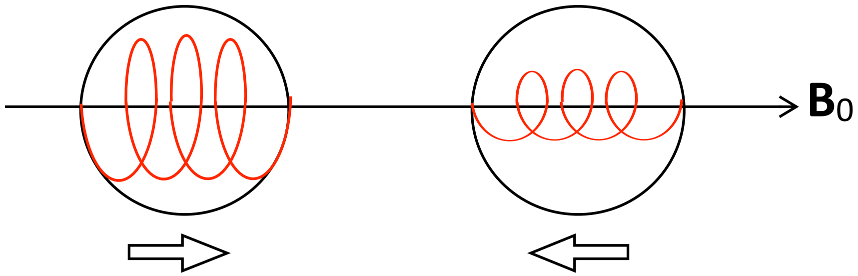


Balance vs. Imbalance

- Balanced : the amplitude (or perturbation) is same.



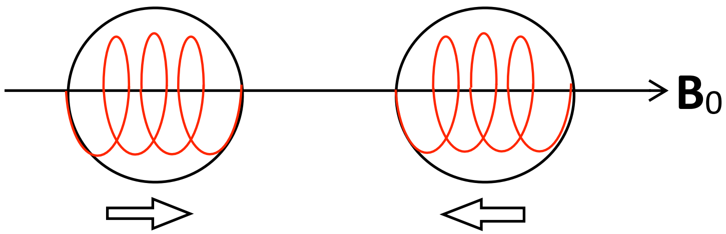
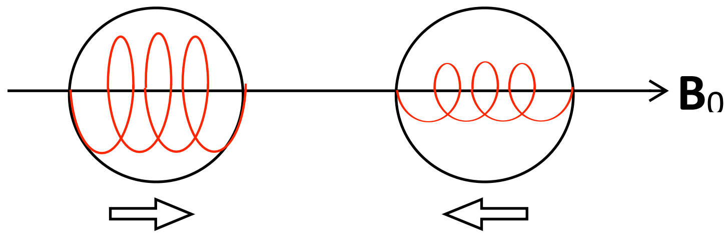
- Imbalanced: the wave packets of different perturbation travel in opposite direction along the magnetic field.



Motivation

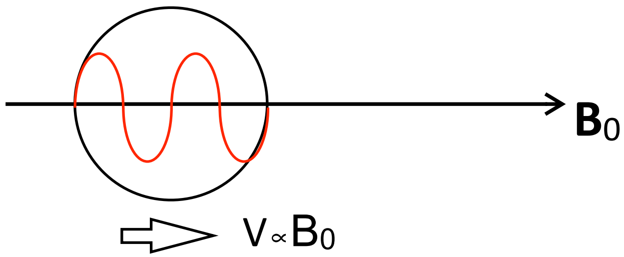
- Imbalanced MHD turbulence
  - Lithwick, Goldreich & Sridhar (2007) : "*Imbalanced Strong MHD Turbulence*"
  - Beresnyak & Lazarian(2008) : "*Strong Imbalanced Turbulence*"
  - Chandran (2008) : "*Strong Anisotropic MHD turbulence with Cross Helicity*"
  - Perez & Boldyrev (2009) : "*Energy and Cross-helicity cascades in Driven MHD turbulence*"
  - Podesta & Bhattacharjee (2009) : "*Theory of incompressible MHD Turbulence with Scale Dependent Alignment and Cross-helicity*"
  - ...

- Imbalanced EMHD : has not been a lot of research.

Balanced EMHD Turbulence	Imbalanced EMHD Turbulence
net helicity = zero	net helicity = not zero
	

Wave packets : MHD vs. EMHD

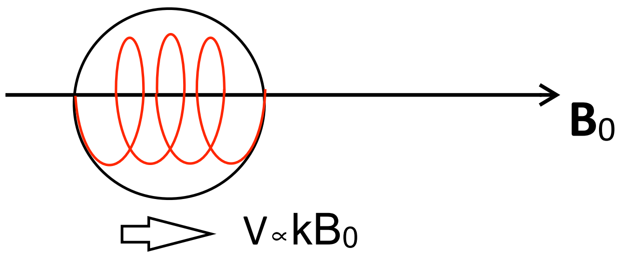
- MHD



- the large eddies pass by small eddies.
  : MHD wave packets is no interaction.

- stable : No change in shape ⇒ No cascade at all

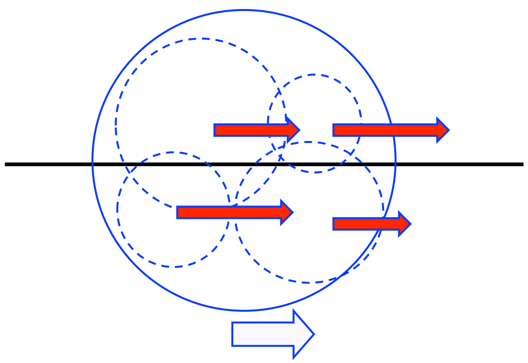
- EMHD



- waves are dispersive
  ⇒ The shape changes in time.
  ⇒ it occurs produces forward+inverse cascade (*Cho 2011*)
  Self-interactions can result in energy cascade

- There is virtually no helicity dissipation because helicity is a conserved quantity.

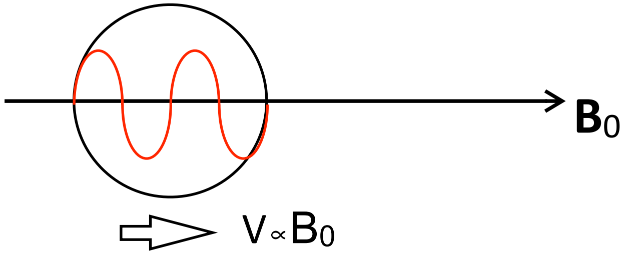
sum of sub-structure



wave packets are sum of many eddies (sub-structure)
-> The velocity of small eddies is faster than the velocity of large eddies.

Wave packets : MHD vs. EMHD

- MHD



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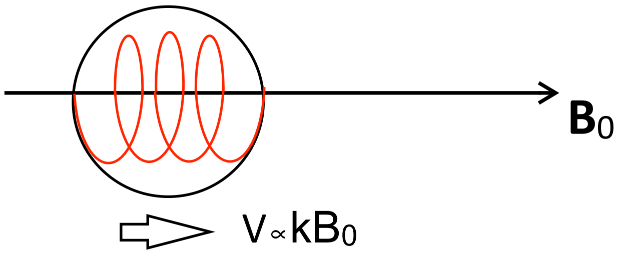
- EMHD



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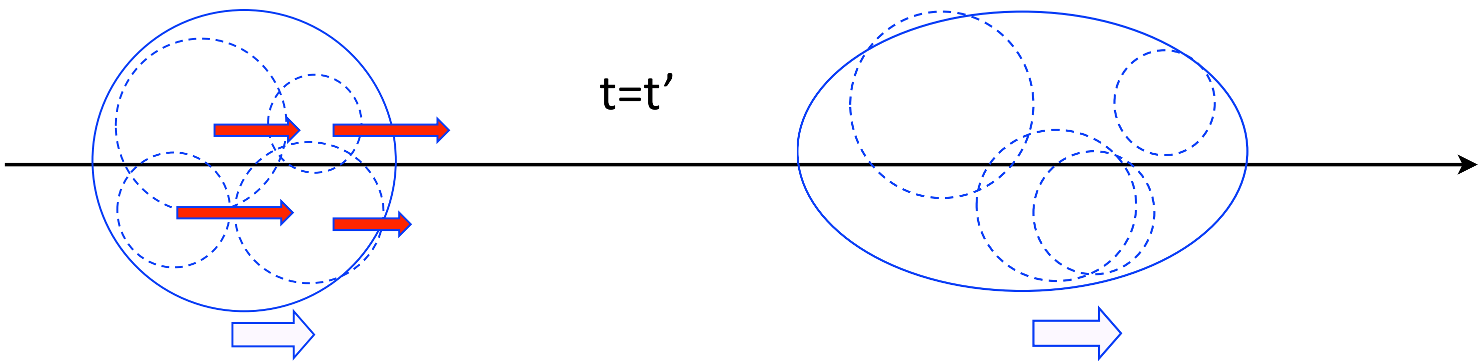
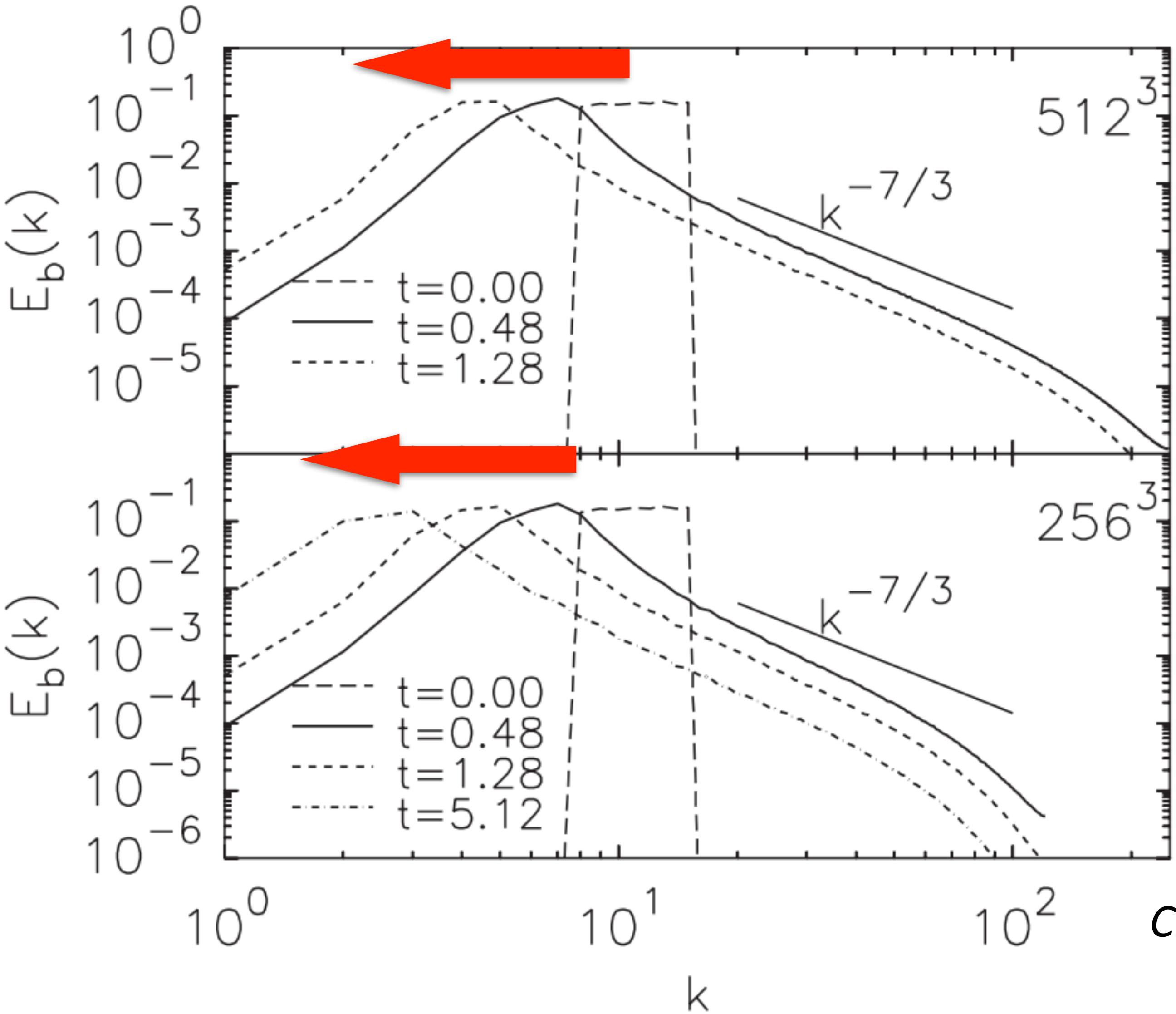
sum of sub-structure

⇒ Self-interactions can result in energy cascade

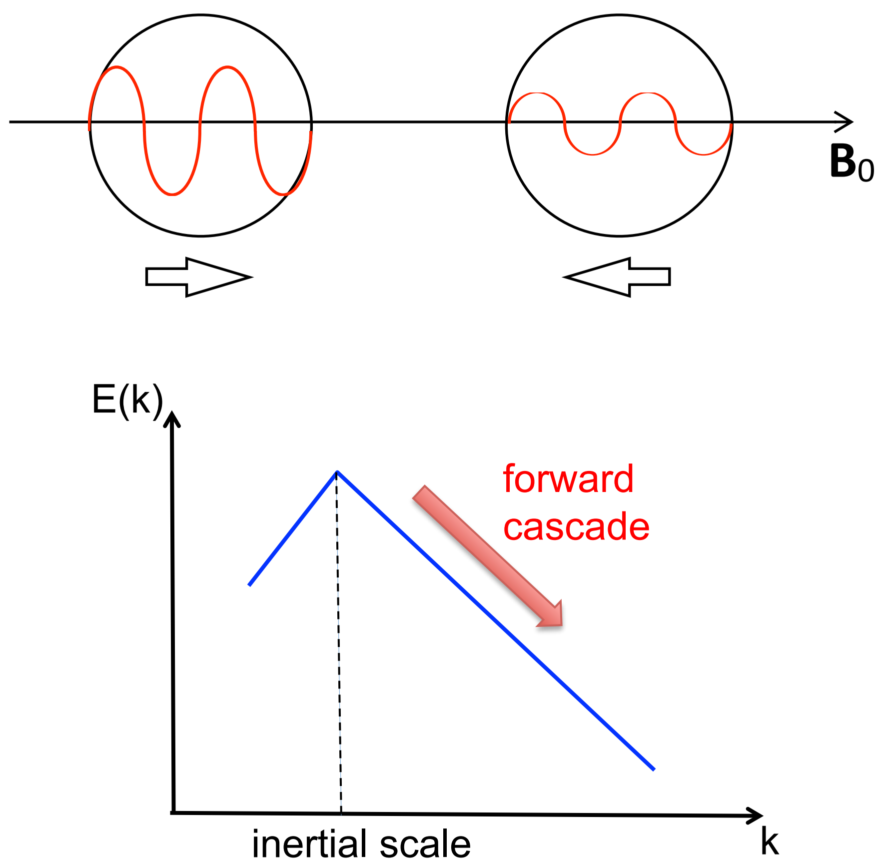


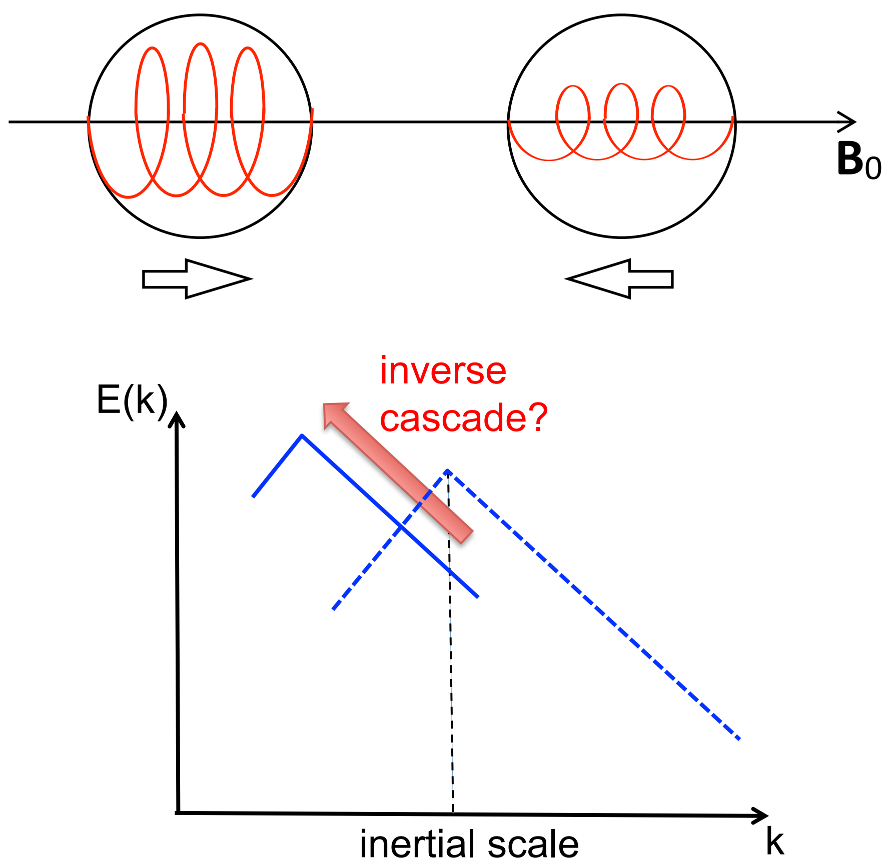
EMHD wave packets



Cho (2011, PRL)

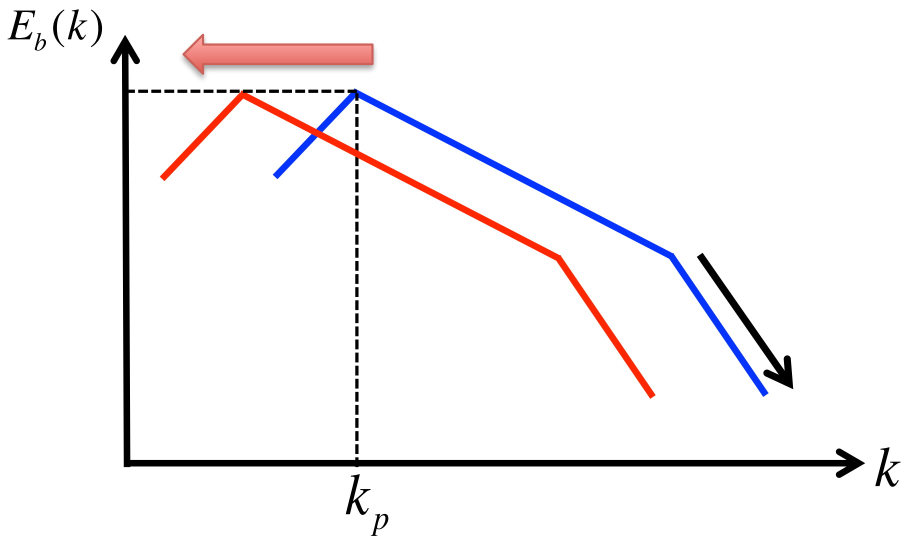
Imbalanced MHD Turbulence	Imbalanced EMHD Turbulence

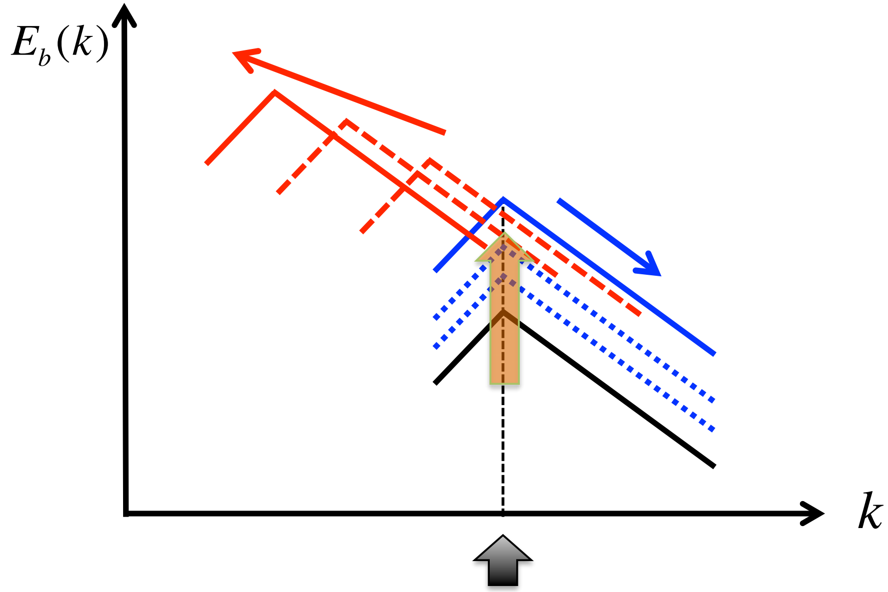




Why occurs inverse cascade?

No driving (Cho 2011)	Magnetic helicity Driving





Magnetic helicity is well conserved quantities !!

$$\int E_h(k)dk \sim \text{constant}$$

$$\int E_h(k)dk \sim \frac{E_b(k_p)}{k_p}k_p \sim E_b(k_p)$$

$$h^2 \sim E_b(k_p) = \text{constant}$$

$$b^2 \sim E_b(k_p)k_p \Rightarrow k_p \downarrow$$

$$h^2 \sim E_b(k_p) \rightarrow \text{accumulated}$$

MHD forward cascade vs. EMHD forward cascade

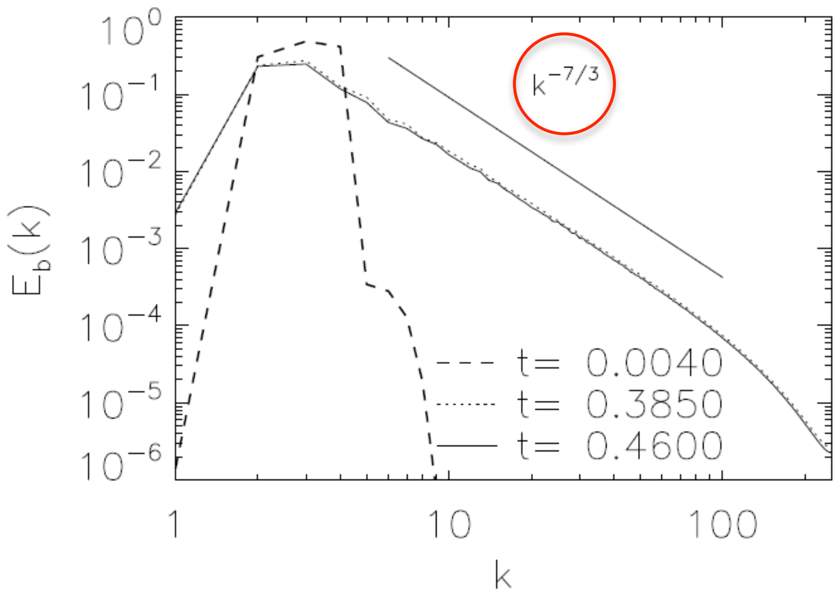
- In strong MHD turbulence, when energy is the cascading quantity, which is the case for scales larger than the energy injection scale, we have

$$\frac{v_l^2}{l/v_l} \sim \frac{b_l^2}{l/b_l} \sim \frac{kE_b(k)}{1/(k\sqrt{kE_b(k)})} = \text{constant} \rightarrow \boxed{E_b(k) \propto k^{-5/3}}$$

- In 3D EMHD turbulence, when magnetic energy is the cascading quantity, we have

$$\frac{b_l^2}{l/v_l} \sim \frac{kE_b}{l/(b_l/l)} \sim \frac{kE_b(k)}{1/(k\sqrt{kk^2E_b(k)})} = \text{constant} \rightarrow \boxed{E_b(k) \propto k^{-7/3}}$$

$\mathbf{v} \propto \nabla \times \mathbf{B}$



(Cho & Lazarian, 2009)

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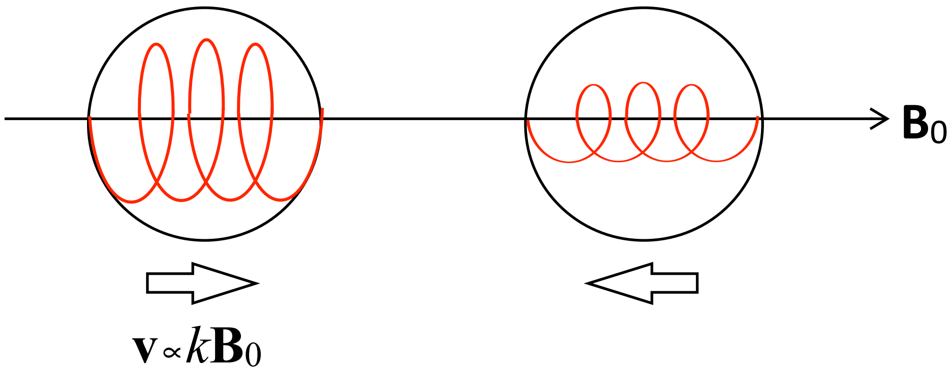
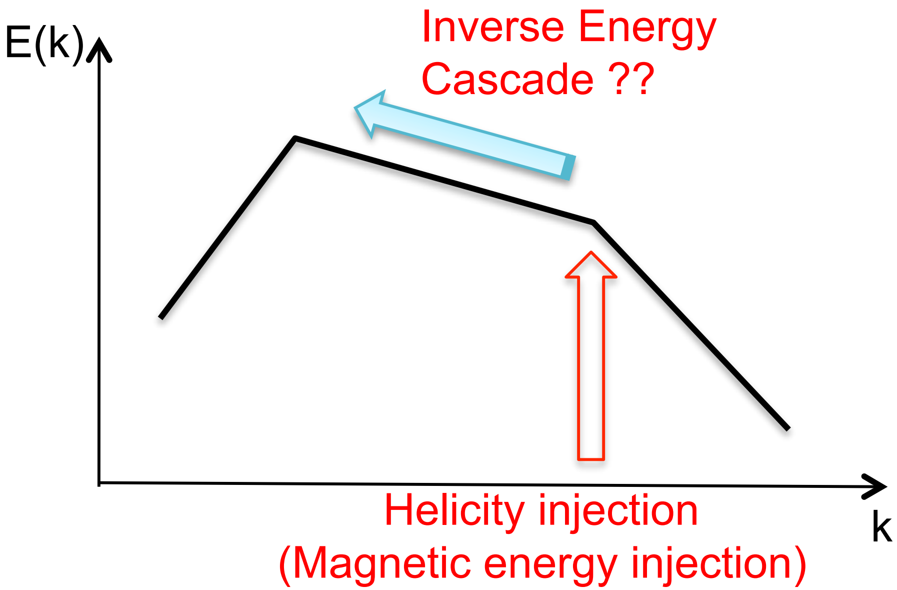
- For an EMHD wave packet
  composed of only '+' wave, the magnetic helicity spectrum, $E_h(k)$ is simply

$$\boxed{E_h(k) = \frac{E_b(k)}{k} \propto k^{-10/3}}$$

Numerical Methods

- We simulated inverse energy cascade in 3D driven imbalanced EMHD turbulence.
  cf.) *Cho 2011* : non-driven EMHD





- In driven-3D EMHD turbulence, when the magnetic helicity is injected at small-scale, we expect inverse energy cascade.

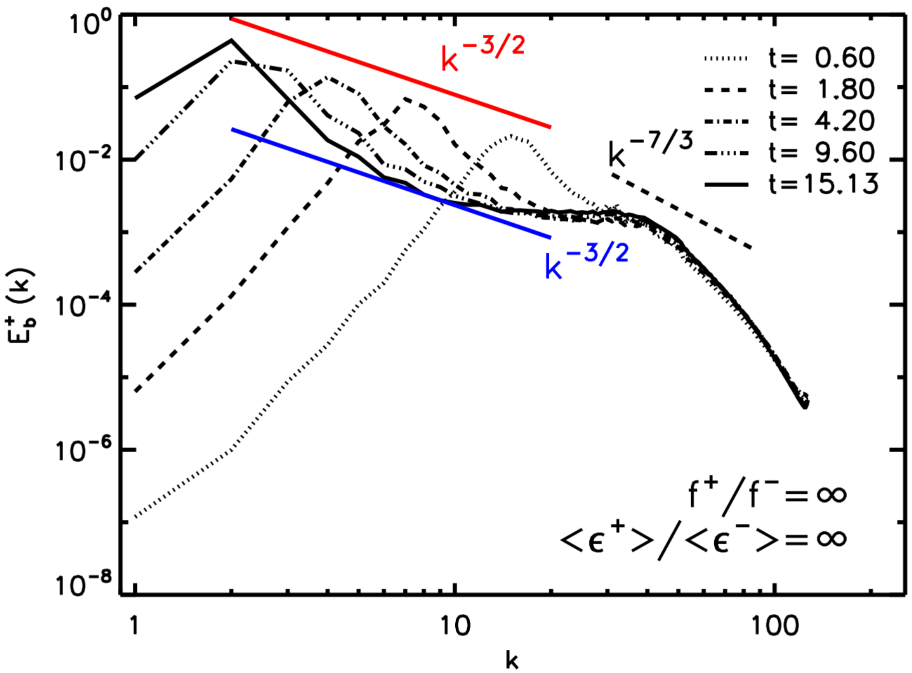
$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times [(\nabla \times \mathbf{B}) \times \mathbf{B}] + \eta \nabla^2 \mathbf{B}$$

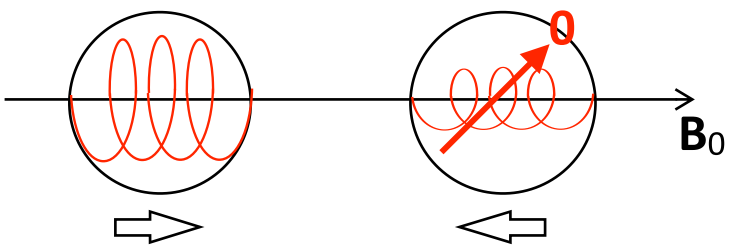
1. Resolution : 256^3
2. Mean magnetic field,
   B_0=1 (Strong magnetized turbulence).
3. Hyper-diffusivity :

$$\eta_h (\nabla^2)^h \mathbf{B}, \quad h = 3$$

Results: Magnetic energy spectra

$\varepsilon^{+} / \varepsilon^{-} = \infty$ (100% imbalanced)





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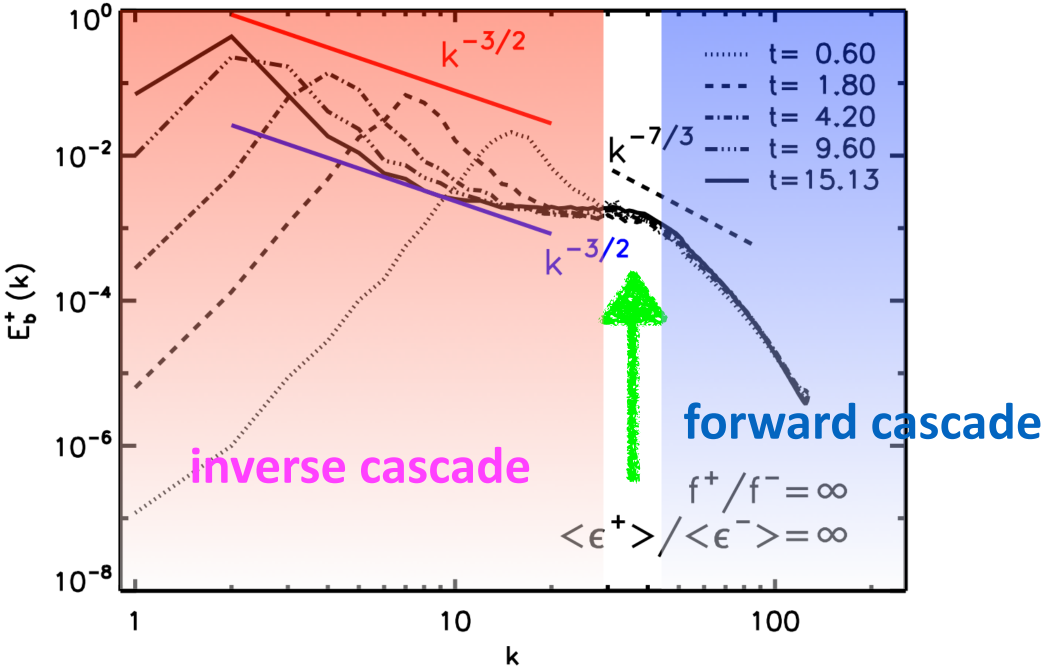

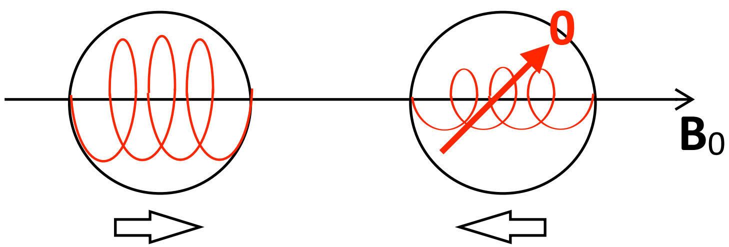

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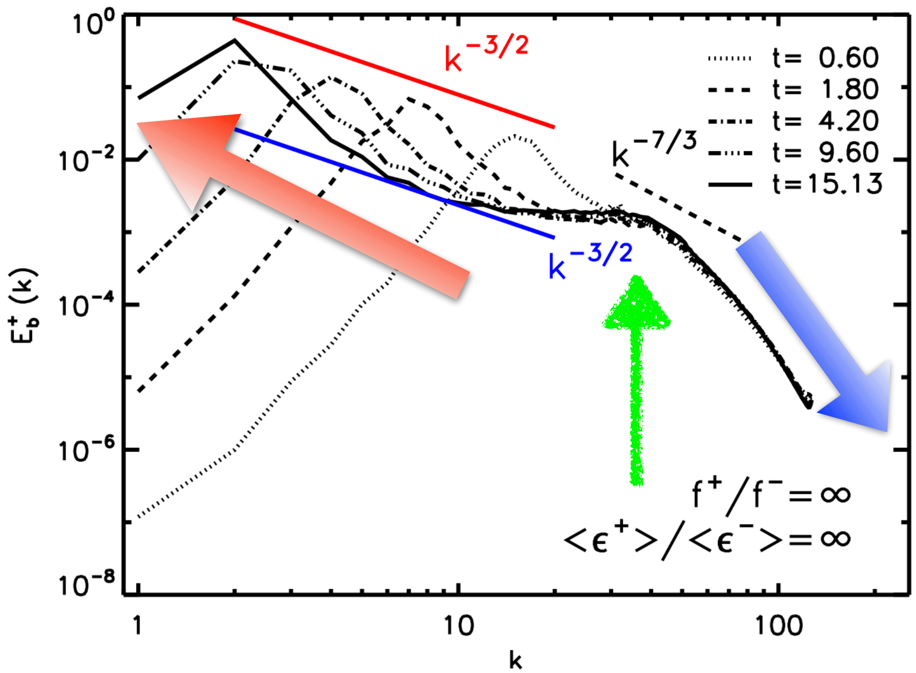
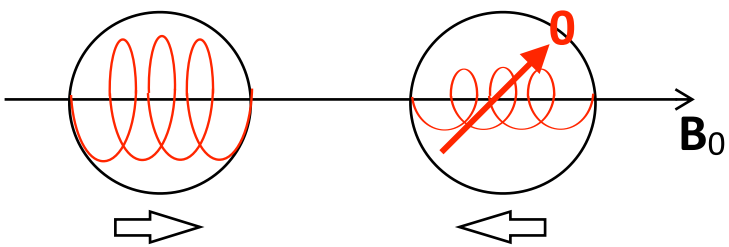




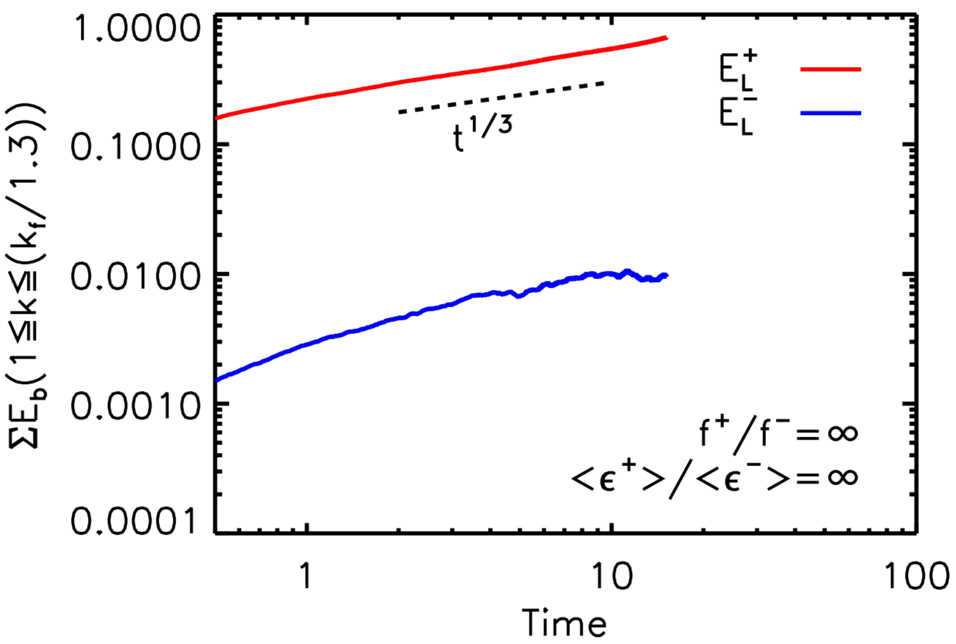
- the magnetic energy is injected
  : initial energy cascade down to smaller scale
  : magnetic energy go up to larger scales
  : peak of k - gradually moves to larger scales.

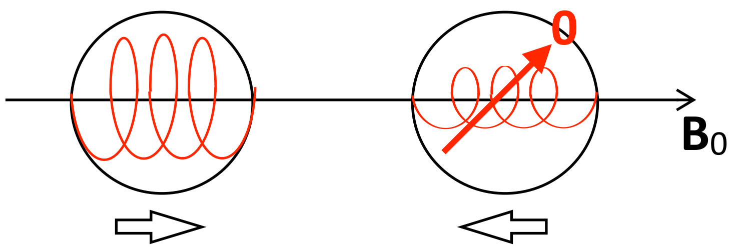
Inverse cascade!!

- $E(k_p) \propto k_p^{-3/2}$

Results: Magnetic energy densities

$\varepsilon^+ / \varepsilon^- = \infty$ (100% imbalanced)

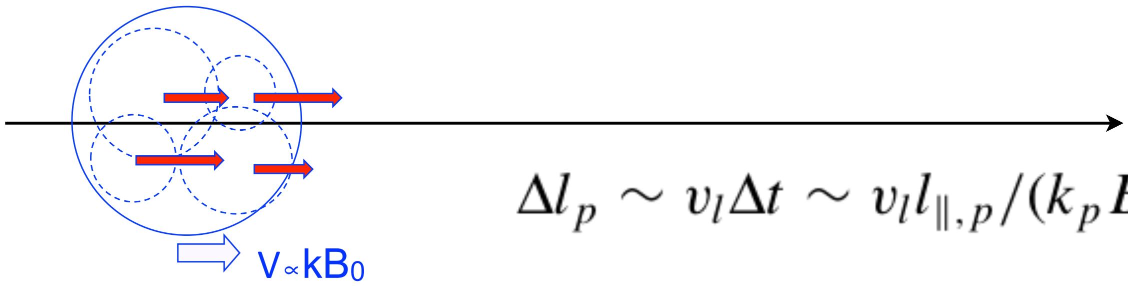




- red solid curves : for dominant waves
  blue solid curves: for sub-dominant waves
- dominant waves : inverse cascade
- sub-dominant waves
  : energy density is not a result of inverse cascade.
  : self-interaction of the large-scale dominant waves.

Spectrum of the Inverse cascade

- The peak of the spectrum at different times follows the law $E(k_p) \propto k_p^{-3/2}$
- Suppose that the inverse cascade is caused by self-interaction of similar-size dominant waves.
- Consider a dominant eddy of size $l_p \sim 1/k_p$.



$$\Delta l_p \sim v_l \Delta t \sim v_l l_{\parallel,p}/(k_p B_0),$$

$\triangle$t = the duration of the interaction.

- the cascade time scale (t_{cas}) + Constancy of helicity transfer

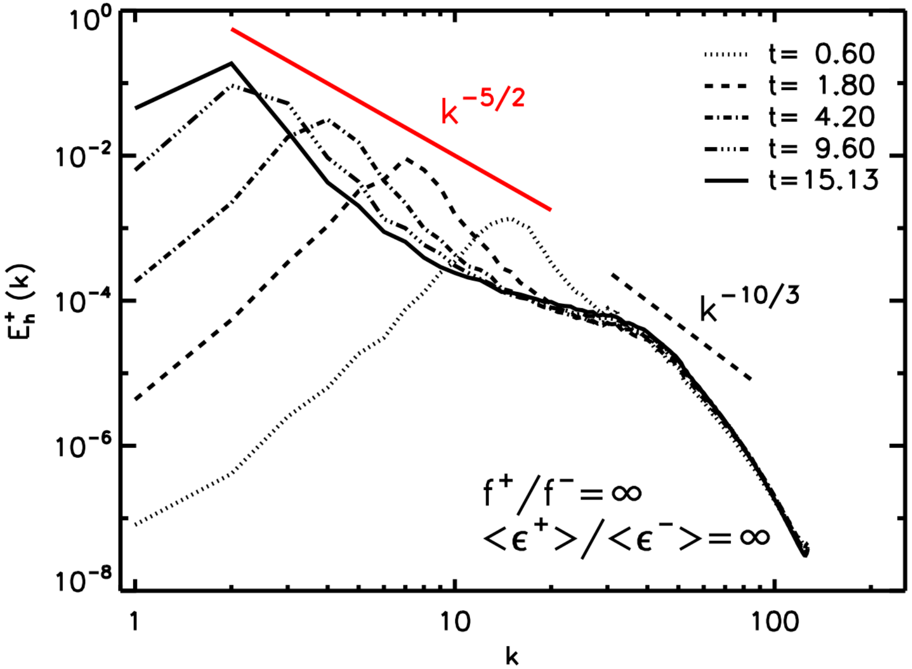
$$t_{\text{cas}} \sim \left(\frac{l_p}{\Delta l_p}\right)^2 \frac{l_{\parallel,p}}{k_p B_0} \quad + \quad \frac{E_b(k_p)}{t_{\text{cas}}} = \text{constant}$$

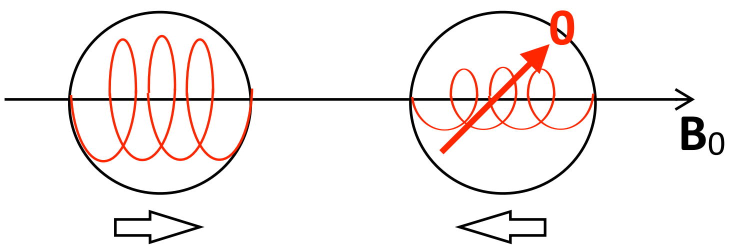
$$\sim \frac{B_0}{k_p^2 b_l^2} \frac{1}{l_{\parallel,p} k_p},$$

$$\Rightarrow \quad E_b(k) \propto \left(\frac{l_p}{l_{\parallel,p}}\right)^{1/2} k_p^{-3/2},$$

Results: Magnetic helicity spectra

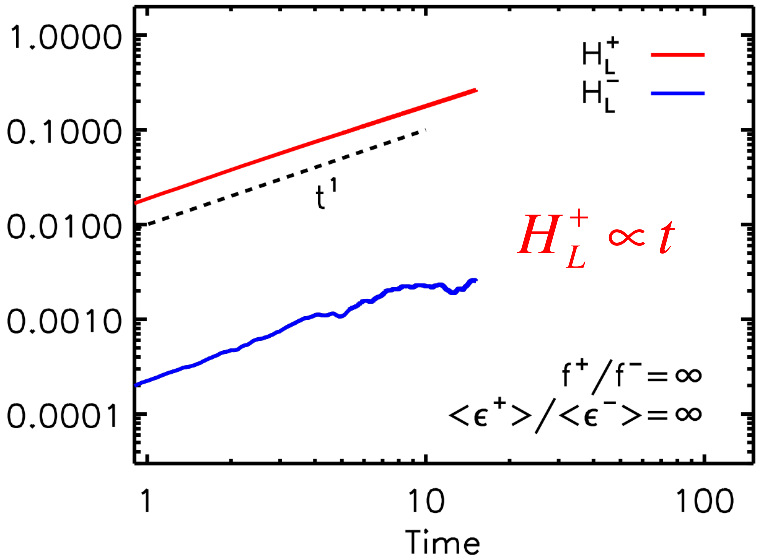
$\varepsilon^{+} / \varepsilon^{-} = \infty$ (100% imbalanced)

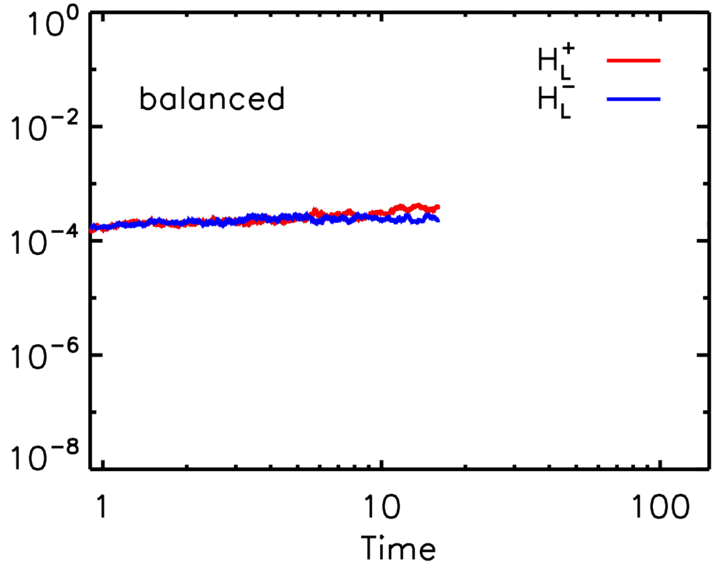


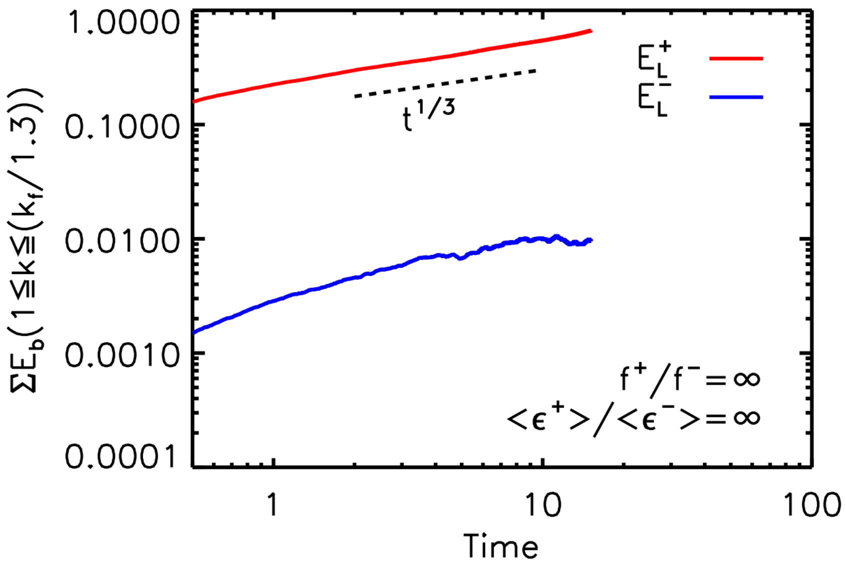


$$E_h(k_p) = \frac{E_b(k_p)}{k_p} \propto k_p^{-5/2}$$

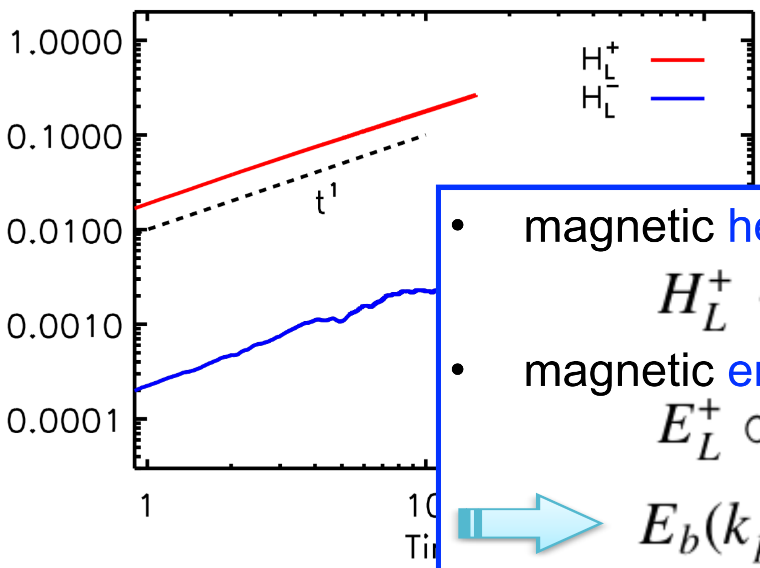
Time Dependence

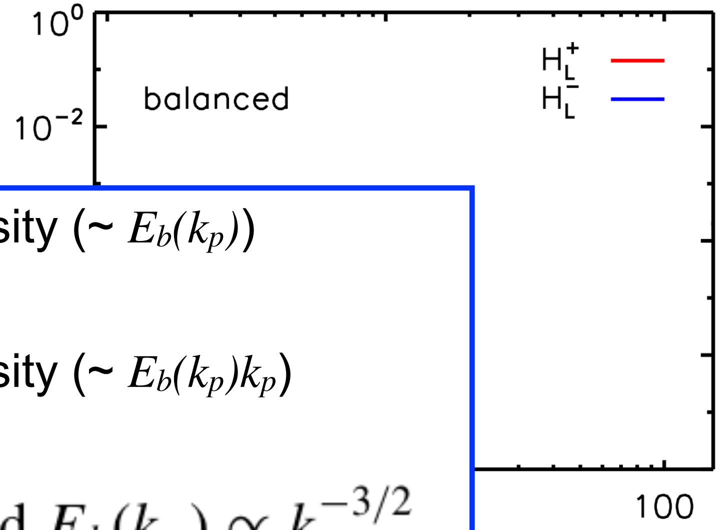






Time Dependence





- magnetic helicity density (~ $E_b(k_p)$)

$$H_L^+ \propto t$$

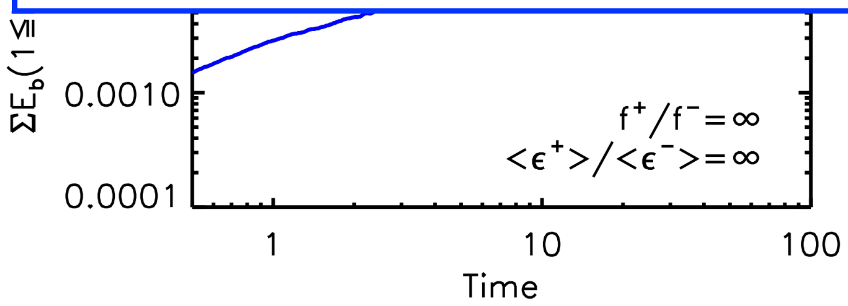
- magnetic energy density (~ $E_b(k_p)k_p$)

$$E_L^+ \propto t^\alpha$$

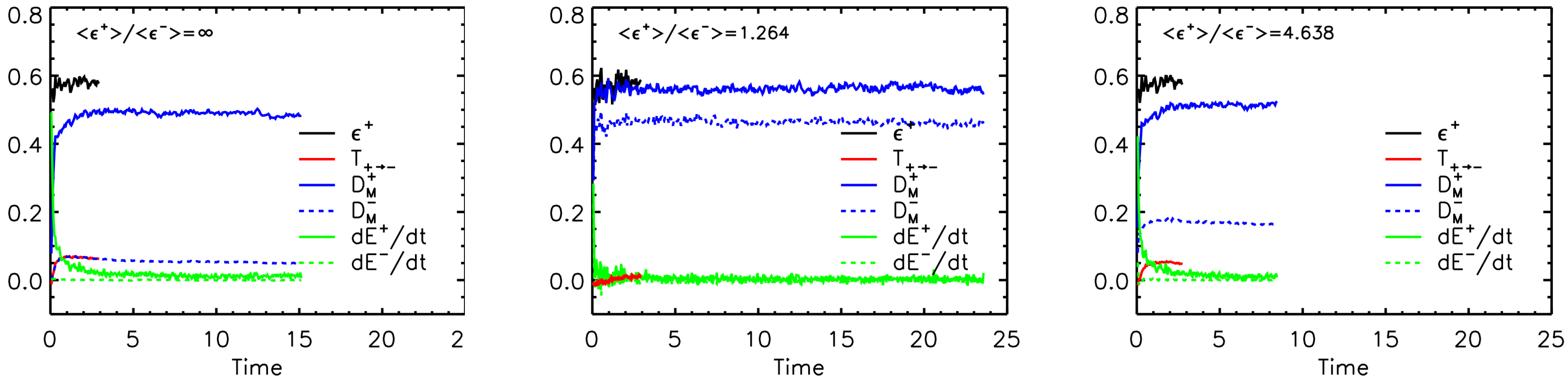
$$E_b(k_p) \propto t \text{ and } E_b(k_p) \propto k_p^{-3/2}$$

$$k_p \propto t^{-2/3}.$$

$$\boxed{E_L^+ \sim E_b(k_p)k_p \propto t^{1/3}.}$$



Energy budget



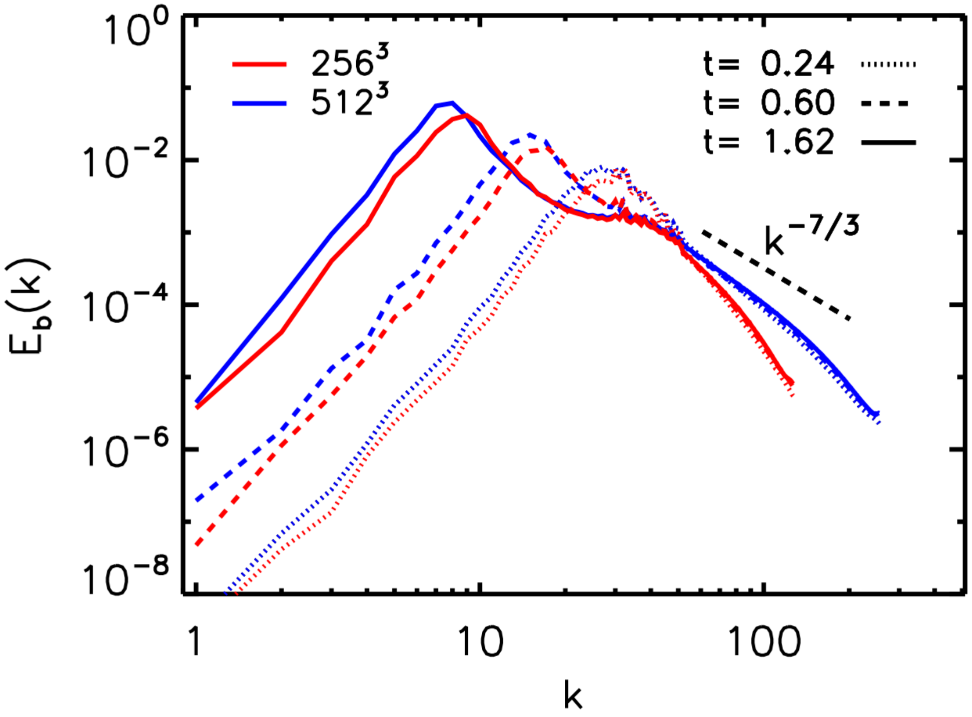
$$\epsilon^+ = \frac{dE_L^+}{dt} + D_M^+ + T_{+\to-},$$

$$\epsilon^- + T_{+\to-} = \frac{dE_L^-}{dt} + D_M^-,$$

- $T_{+\to-}$: the rate of energy transfer from the dominant waves to the sub-dominant waves.
- the degree of imbalance is 100% -> $T_{+\to-} / \varepsilon$ is relatively large.

Effect of Resolution

$\varepsilon^{+} / \varepsilon^{-} \approx 4.571$



- Do not fully resolve the inertial range for forward cascade.
- Could it be a problem?
  - it may not be a problem.
  - main concern: investigate inverse cascade
- Change of the $E_b(k)$: 512^3 is faster than 256^3.
  - However, overall behavior is similar.

Summary

- Imbalanced EMHD turbulence clearly shows inverse cascade.
- The spectral behavior of the magnetic energy density on scales larger than the driving scale is different from that of other types of turbulence.
- The large-scale magnetic field does not show a well defined single power-law spectrum, indicating that inverse cascade is not a self-similar process.
- Instead, the peak of the spectrum follows a power-law close to $k^{-3/2}$.
- We can explain the behavior using self-interaction of EMHD waves moving in one direction.
- Self-interaction of EMHD waves moving in one direction can generate opposite-traveling waves.
- Therefore, even in a strongly imbalanced EMHD turbulence, we can observe a certain amount of opposite-traveling waves.

THANK YOU !!!