Numerical Study of Compressible Isothermal Magnetohydrodynamic Turbulence

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Astrophysical shocks

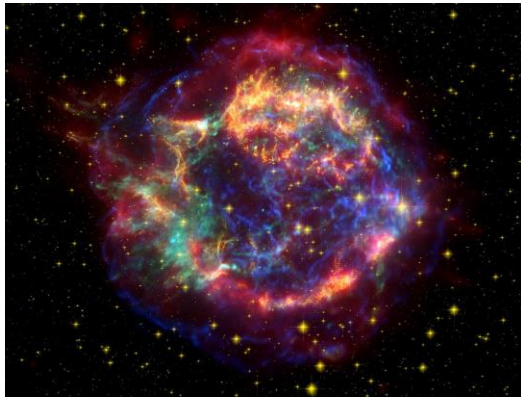
- Shock waves are common in astrophysical environments
  - Supernova remnant shock waves

    Deposit energy and high metallicity plasma into ISM

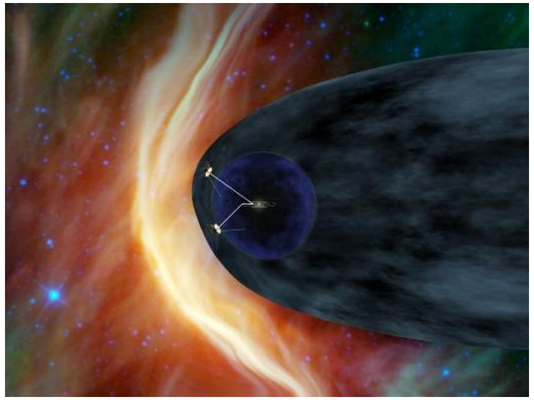
    Instigate star formation
  - Heliospheric shocks

    Frequently driven by solar eruptions

    Routinely observed at 1 AU
  - Accretion shocks in clusters of galaxies
  - Shocks in star formation regions
  - Gamma ray bursts, AGN jets, pulsar wind nebulae, etc.



Cassiopeia A

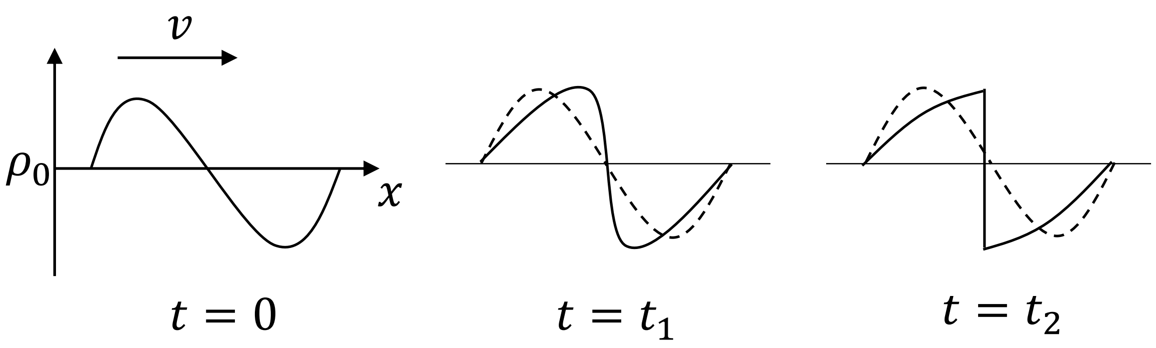


Voyagers in the Heliosheath

Understanding how such shock structures form and their statistics in turbulent media is importance in astrophysics.

Shock waves

When a wave moves faster than the local speed of sound in a fluid, shock waves are generated.



(dashed line wave profile for c_s constant)

- Density perturbation moving to right at $t = 0$
- At $t = t_1$ the crest has begun to overtake the trough
- shock forms at $t = t_2$
- Mach number $M = v/c_s$

- In the absence of magnetic fields, information travels with sound speed, c_s ($M < 1$ subsonic, $M > 1$ supersonic)
- If magnetic field is present, disturbances will travel along B at Alfvén speed $v_A = B/\sqrt{4\pi\rho}$
- MHD supports three basic wave modes:
  - Alfven wave (non-compressive)
  - Fast and slow MHD waves (compressive)

Purpose of Study

1. We studied MHD turbulence in a variety of astrophysical environments by performing numerical simulations of isothermal, compressible MHD turbulence with resolution up to 512^3.
2. The main goal is to identify the formulas for fast and slow Mach numbers and understand the characteristic properties of fast and slow MHD shocks.

Contents

- MHD equations of compressible, isothermal gas
  - simulation initial conditions
- Formula for Fast and Slow Mach numbers
- Simulation results
  - fast and slow Mach number 2D structures
  - fast and slow Mach number distribution
- Conclusion and Future work

MHD equations of compressible, isothermal gas

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \qquad \textit{CONTINUITY}$$

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{c_s^2}{\rho} \nabla \rho - \frac{1}{\rho} (\nabla \times \mathbf{B}) \times \mathbf{B} = 0 \qquad \textit{MOMENTUM}$$

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0 \qquad \textit{INDUCTION}$$

with the additional constraint for the absence of magnetic monopoles, $\nabla \cdot \mathbf{B} = 0$

Simulation initial conditions

- Isothermal TVD Code (Kim, et al. 1998)
- Turbulence was driven by solenoidal ($\nabla \cdot \boldsymbol{\delta} \mathbf{v} = 0$) forcing (Stone et al. 1999; Mac Low 1999)
- Initial conditions:
  - computational domain $L_0 = L_x = L_y = L_z = 1$ with periodic boundaries
  - velocity forcing drawn from Gaussian random field, $P_k \propto k^6 \exp(-8k/k_{peak})$, where $k_{peak} = 4\pi/L_0$
  - $\rho_0 = 1$ and $B_0 = \left(\sqrt{2/\beta_0}, 0,0\right)$ with $\beta_0 = P_g/P_B = 0.1, 1, 10$
  - $M_{\rm turb} = v_{\rm rms}/c_s = 0.5, 1, 2, 4\ , 7$
  - resolution: 512^3 grid cell

MHD equations of compressible, isothermal gas

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$ *CONTINUITY*

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} + \frac{c_s^2}{\rho} \nabla \rho - \frac{1}{\rho} (\nabla \times \mathbf{B}) \times \mathbf{B} = 0$$ *MOMENTUM*

$$\frac{\partial \mathbf{B}}{\partial t} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0$$ *INDUCTION*

with the additional constraint for the absence of magnetic monopoles, $\nabla \cdot \mathbf{B} = 0$

Simulation initial conditions

- saturation was reached $\sim 2t_d$ where $t_\mathrm{d} = L_0/(2M_\mathrm{turb}c_s)$

- Comparison of observation & simulation :
  - cloud clump of size $L_0 = 2\mathrm{pc}$, $T = 10\mathrm{K}$, $\mathrm{c_s} \approx 0.2\mathrm{km/s}$, $n_\mathrm{H2} = 10^3 \mathrm{cm}^{-3}$ (observed MCs)
  - sound crossing time: $t_\mathrm{s} = L_0/c_s \sim 10\mathrm{Myr}$,
    dynamical time $t_\mathrm{d} = L_0/(2M_\mathrm{turb}c_s) = 5\mathrm{Myr}/M_\mathrm{turb}$
  - $B_0 = 1.4\mu\mathrm{G}\beta^{-1/2}(\mathrm{T}/10\mathrm{K})^{1/2}(n_\mathrm{H2}/10^2\mathrm{cm}^{-3})^{1/2} \sim 14,\ 4.4,\ 1.4\ \mu\mathrm{G}$ for $\beta = 0.1, 1, 10$

MHD shock

- We will work in a frame where shock is stationary (*shock fame*)
- *x*-axis will be a aligned with the shock normal,

  so plane of shock is parallel to *yz*-plane
- The jump across the any quantities of X can be expressed using following notation:

$$[X] = X_{\text{pre}} - X_{\text{post}} = X_1 - X_2$$

- MHD jump relation is given us a set of conservation equations

$$\partial Q / \partial t + \nabla \cdot \boldsymbol{F} = 0$$

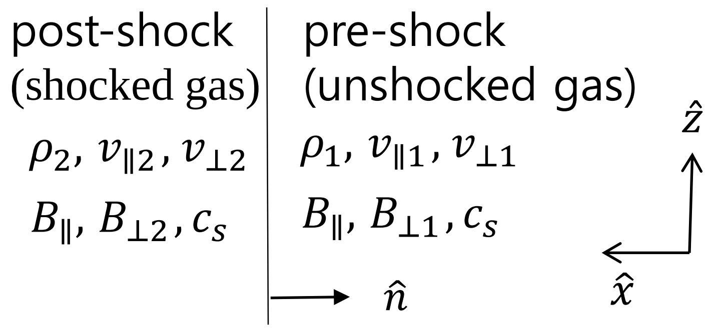
(Q : conserved quantities, $\boldsymbol{F}$: flux)

- If shock is steady ($\partial / \partial t = 0$) and 1D ($\partial / \partial y = \partial / \partial z = 0$),

$$\partial F_x / \partial x = 0$$

- Which is implies that $(F_1 - F_2) \cdot \hat{n} = 0$ ⇨ $[F_n] = 0$

plane-parallel shock



$\parallel$: parallel to shock normal
$\perp$: perpendicular to shock normal
$B_{\parallel}, c_s$: same in pre-shock and Post-shock

Formula for Fast and Slow Mach numbers

Shock jump conditions of both sides of the shock front are as follows:

$$\rho_1 v_{\parallel 1} = \rho_2 v_{\parallel 2} \quad (1)$$

$$\rho_1 v_{\parallel 1}^2 + c_s^2 \rho_1 + \frac{1}{2} B_{\perp 1}^2 = \rho_2 v_{\parallel 2}^2 + c_s^2 \rho_2 + \frac{1}{2} B_{\perp 2}^2 \quad (2)$$

$$\rho_1 v_{\parallel 1} v_{\perp 1} - B_{\parallel} B_{\perp 1} = \rho_2 v_{\parallel 2} v_{\perp 2} - B_{\parallel} B_{\perp 2} \quad (3)$$

$$v_{\parallel 1} B_{\perp 1} - v_{\perp 1} B_{\parallel} = v_{\parallel 2} B_{\perp 2} - v_{\perp 2} B_{\parallel} \quad (4)$$

$$\chi = \frac{\rho_2}{\rho_1} = \frac{v_{\parallel 1}}{v_{\parallel 2}}$$

$$\rho_1 v_{\parallel 1}^2 + c_s^2 \rho_1 + \frac{1}{2} B_{\perp 1}^2 = \frac{\rho_1 v_{\parallel 1}^2}{\chi} + \chi c_s^2 \rho_1 + \frac{1}{2} B_{\perp 2}^2 \quad (5)$$

$$\left(\rho_1 v_{\parallel 1}^2 - B_{\parallel}^2\right) B_{\perp 1} = \left(\frac{\rho_1 v_{\parallel 1}^2}{\chi} - B_{\parallel}^2\right) B_{\perp 2} \quad (6)$$

$B_{\perp 2}$ substituted into the eq 5, and set $M_s = v_{\parallel 1}/c_s, c_{A\parallel} = B_{\parallel}/\sqrt{\rho_1}, c_{A\perp} = B_{\perp 1}/\sqrt{\rho_1}, \alpha_{\parallel} = c_{A\parallel}/c_s, \alpha_{\perp} = c_{A\perp}/c_s$

$$M_s^6 - \left[\left(1 + 2\alpha_{\parallel}^2 + \frac{1}{2}\alpha_{\perp}^2\right)\chi + \frac{1}{2}\alpha_{\perp}^2 \chi^2\right] M_s^4 + \alpha_{\parallel}^2 \left(2 + \alpha_{\parallel}^2 + \alpha_{\perp}^2\right)\chi^2 M_s^2 - \alpha_{\parallel}^4 \chi^3 = 0 \quad (7)$$

the phase velocity of fast and slow modes is derived from eq 7 with $\chi = 1$ as follows:

$$\frac{c_{\mathrm{fa,sl}}^2}{c_s^2} = \frac{1}{2}\left[1 + \alpha_{\parallel}^2 + \alpha_{\perp}^2 \pm \sqrt{\left(1 + \alpha_{\parallel}^2 + \alpha_{\perp}^2\right)^2 - 4\alpha_{\parallel}^2}\right] \quad (8)$$

Fast and Slow Mach numbers

$$M_{\mathrm{fa}} = M_{\mathrm{s}} c_{\mathrm{s}}/c_{\mathrm{fa}}, \quad M_{\mathrm{sl}} = M_{\mathrm{s}} c_{\mathrm{s}}/c_{\mathrm{sl}}$$

How to find M_s from 6th order equation?

let $x = M_s^2$ $\quad x^3 + C_a x^2 + C_b x^2 + C_c = 0$

where $C_a = -\left[\left(1 + 2\alpha_\parallel^2 + \frac{1}{2}\alpha_\perp^2\right)\chi + \frac{1}{2}\alpha_\perp^2\chi^2\right]$ $\quad C_b = \alpha_\parallel^2(2 + \alpha_\parallel^2 + \alpha_\perp^2)\chi^2$ $\quad C_c = -\alpha_\parallel^4\chi^3$

Compute $Q = (C_a^2 - 3C_b)/9$, $R = (2C_a^3 - 9C_aC_b + 27C_C)/54$

If $Q^2 < R^3$, then the cubic equation has three real roots (Press et al. 1986) as follows:

$x_1 = -2\sqrt{Q}\cos\left(\frac{\theta}{3}\right) - \frac{C_a}{3}$ $\quad x_2 = -2\sqrt{Q}\cos\left(\frac{\theta + 2\pi}{3}\right) - \frac{C_a}{3}$ $\quad x_3 = -2\sqrt{Q}\cos\left(\frac{\theta - 2\pi}{3}\right) - \frac{C_a}{3}$ $\quad$ where $\theta = \arccos\left(R/\sqrt{Q^3}\right)$

Otherwise $Q^2 > R^3$, the cubic equation has only one real root.

Compute $A = -\mathrm{sgn}(R)\left[|R| + \sqrt{R^2 - Q^3}\right]^{\frac{1}{3}}$, $B = Q/A\ (A \neq 0)$ or $0\ (A = 0)$ and calculate the real root as follows:

$x_1 = (A + B) - C_a/3$

How to find M_s from 6th order equation?

let $x = M_s^2$ $\quad x^3 + C_a x^2 + C_b x^2 + C_c = 0$

where $C_a = -\left[\left(1 + 2\alpha_\parallel^2 + \frac{1}{2}\alpha_\perp^2\right)\chi + \frac{1}{2}\alpha_\perp^2\chi^2\right]$ $\quad C_b = \alpha_\parallel^2(2 + \alpha_\parallel^2 + \alpha_\perp^2)\chi^2$ $\quad C_c = -\alpha_\parallel^4\chi^3$

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If $Q^2 < R^3$, then the cubic equation has three real roots (Press et al. 1986) as follows:

$$x_1 = -2\sqrt{Q}\cos\left(\frac{\theta}{3}\right) - \frac{C_a}{3}$$

slow mode

$$x_2 = -2\sqrt{Q}\cos\left(\frac{\theta + 2\pi}{3}\right) - \frac{C_a}{3}$$

fast mode

$$x_3 = -2\sqrt{Q}\cos\left(\frac{\theta - 2\pi}{3}\right) - \frac{C_a}{3}$$

where $\theta = \arccos\left(R/\sqrt{Q^3}\right)$

Otherwise $Q^2 > R^3$, the cubic equation has only one real root.

Compute $A = -\mathrm{sgn}(R)\left[|R| + \sqrt{R^2 - Q^3}\right]^{\frac{1}{3}}$, $B = Q/A\ (A \neq 0)$ or $0\ (A = 0)$ and calculate the real root as follows:

$$x_1 = (A + B) - C_a/3$$

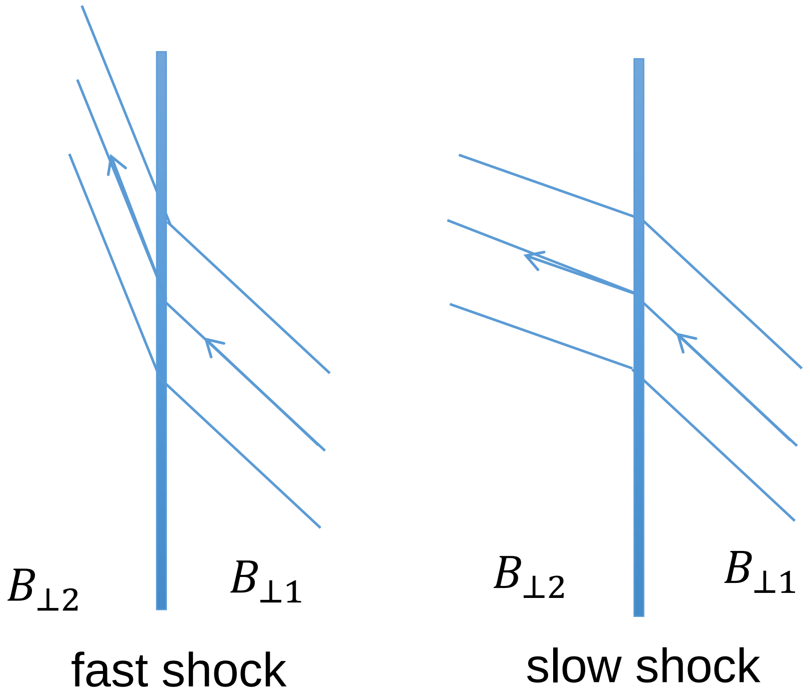
fast mode

$$0$$

slow mode

Properties of Fast and Slow shocks:

- Shock wave: flow crosses surface of discontinuity accompanied by compression and dissipation



1. They are compressive, with $\chi > 1$, which implies the $P_2 > P_1$

2. $v_{\parallel 1}$ exceeds the fast/slow speed ahead the shock $(M_{\rm fa}, M_{\rm sl} > 1)$

3. $B_{\parallel}$ remains unchanged over the shock

4. From Jump conditions

$$\frac{B_{\perp 2}}{B_{\perp 1}} = \frac{\left(\rho_1 v_{\parallel 1}^2 - B_{\parallel}^2\right)}{\left(\rho_1 v_{\parallel 1}^2/\chi - B_{\parallel}^2\right)} = \chi \frac{\left(M_s^2 - \alpha_{\parallel}^2\right)}{\left(M_s^2 - \alpha_{\parallel}^2 \chi\right)}$$

5. When $M_s^2 > \alpha_{\parallel}^2 \chi (> \alpha_{\parallel}^2)$, this equation implies that $B_{\perp 2}/B_{\perp 1} > 1$, called a fast shock.

6. When $\left(\alpha_{\parallel}^2 \chi >\right) \alpha_{\parallel}^2 > M_s^2$, this equation implies that $B_{\perp 2}/B_{\perp 1} < 1$, called a slow shock.

Shock family criteria

(1) Find shock

- Shock cells: $\nabla \cdot \mathbf{v} < 0$ and $\chi > 1.005^2$

- Shock center: $\nabla \cdot \mathbf{v}$ is local minimum and labeled this center as part of a shock surface.

- Shock quantities: $\max(\rho_{i+2}, \rho_{i-2})$ is post-shock and $\min(\rho_{i+2}, \rho_{i-2})$ is pre-shock

(2) Find Fast and Slow shocks

- Fast shock

$$B_{\perp 2}/B_{\perp 1} \geq 1,\ M_{\mathrm{fa}} \geq 1.03,\ \chi \geq 1.06$$

- Slow shock

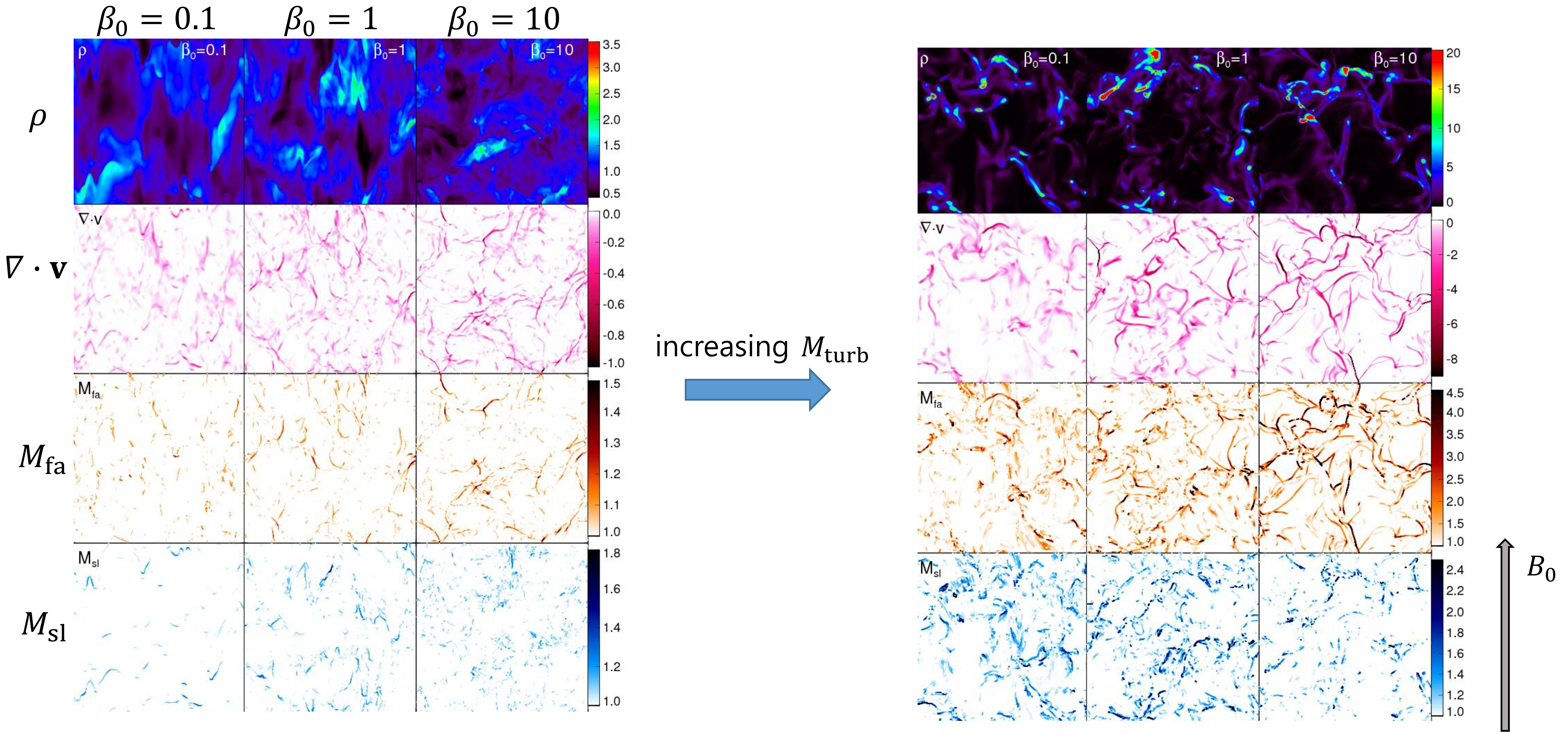
$$0 < B_{\perp 2}/B_{\perp 1} < 1,\ 1.03 \leq M_{\mathrm{sl}} < c_{\mathrm{A}\parallel}/c_{\mathrm{sl}},\ 1.06 \leq \chi < \left[\left(2\alpha_\parallel^2 + \alpha_\perp^2 + 2\right) + \sqrt{\left(2\alpha_\parallel^2 + \alpha_\perp^2 + 2\right)^2 - 16\alpha_\parallel^2}\right]/4$$

$$\rho_1 v_{\parallel 1}^2 + c_s^2 \rho_1 + \frac{1}{2} B_{\perp 1}^2 = \frac{\rho_1 v_{\parallel 1}^2}{\chi} + \chi c_s^2 \rho_1 + \frac{1}{2} B_{\perp 2}^2 \qquad (5)$$

$$\left(\rho_1 v_{\parallel 1}^2 - B_\parallel^2\right) B_{\perp 1} = \left(\frac{\rho_1 v_{\parallel 1}^2}{\chi} - B_\parallel^2\right) B_{\perp 2} \qquad (6)$$

with $B_{\perp 2} = 0$

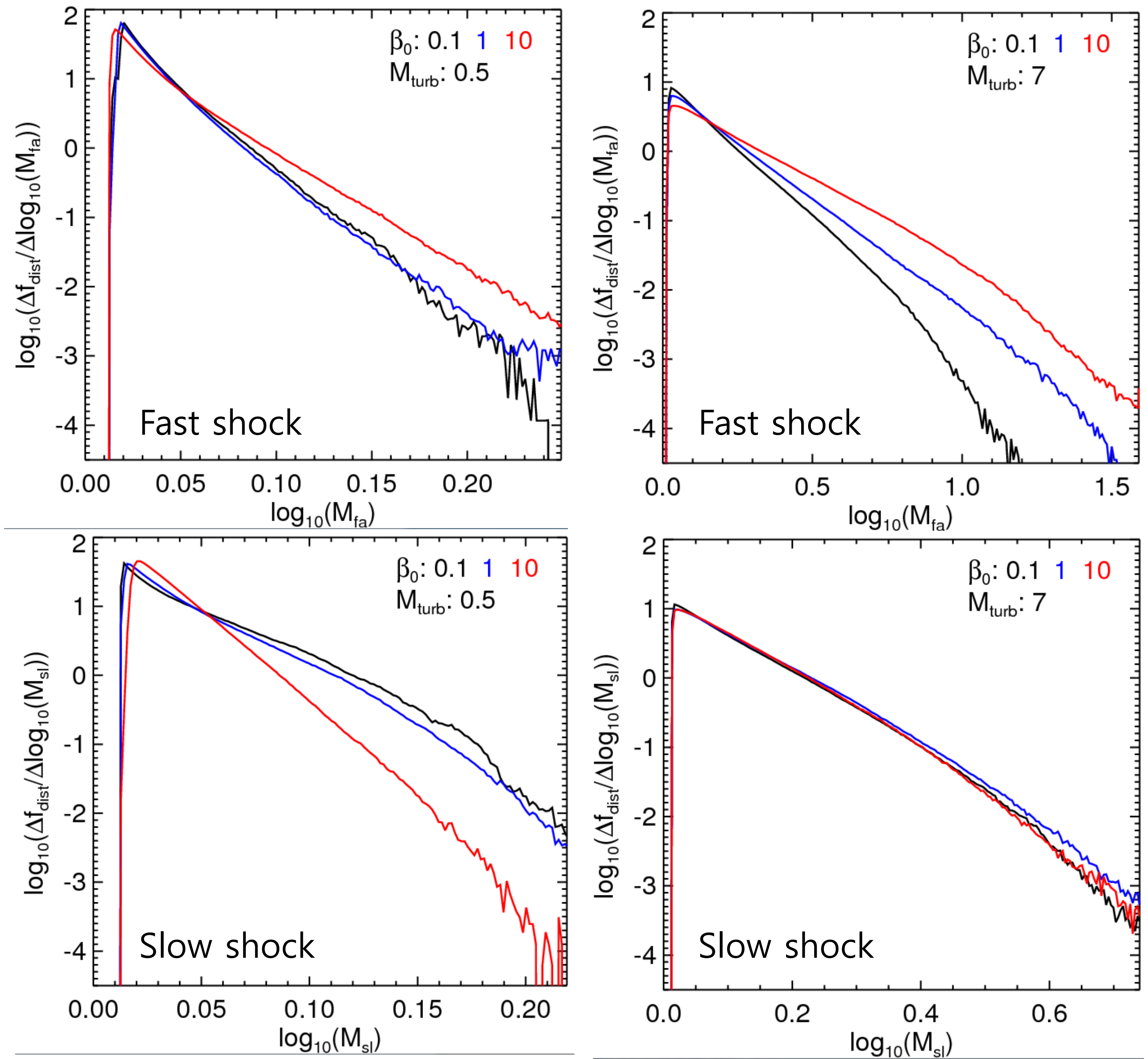
Visualization of the physical quantities of MHD simulations $(t/t_d = 4)$



$(M_{turb} = 0.5)$ $(M_{turb} = 7)$

- Filaments and sheets with high density are formed in a flow with higher M_{turb}
- Supersonic turbulence → Dense core formation and shocks ↑

Distribution of fast and slow shocks $(t/t_d = 3 \sim 4)$



- The weaker shocks dominate the shock number distribution (Smith et al. 2000; Lehmann et al. 2016)

- M_{turb} increases the slope becomes shallower

- β_0 increases fast shocks become dominated, slow shocks become undominated

$$\frac{B_{\perp 2}}{B_{\perp 1}} = \frac{\left(\rho_1 v_{\parallel 1}^2 - B_{\parallel}^2\right)}{\left(\rho_1 v_{\parallel 1}^2/\chi - B_{\parallel}^2\right)} \quad \begin{array}{ll} > 1 & \text{fast shock} \\ < 1 & \text{slow shock} \end{array}$$

Conclusion

- We studied compressible isothermal MHD turbulence in a variety of M_{turb} and β_0, with resolution 512^3
- Filaments and sheets with high density are formed in a flow with higher M_{turb} and there are associated with strongly converging region.
- The distribution of fast and slow shocks
  - M_{turb} increases the slope becomes shallower
  - β_0 increases fast shocks become dominated
    slow shocks become undominated

Future work

- We need to do more simulation (up to 1024^3) to analyze the results according to the resolution.
- No physical dissipation terms are modelled, but numerical diffusion is unavoidable.
- In saturation

  Energy injection = Energy dissipation at shocks + Energy dissipation due to numerical viscosity & resistivity