MHD Wave Analysis

in Cartesian and Cylindrical Coordinates

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1. Cartesian Coordinate

MHD equations

Primitive forms

density

velocity

magnetic induction

adiabatic index

Gauss law

pressure

1. Cartesian Coordinate

MHD equations

1D $\vec{\nabla} \rightarrow \frac{\partial}{\partial x}\hat{i}$

density $\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u_x) = 0$

velocity $\frac{\partial \vec{u}}{\partial t} + u_x \frac{\partial \vec{u}}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x}\hat{i} - \frac{1}{\rho}\left(\frac{\partial}{\partial x}\hat{i} \times \vec{B}\right) \times \vec{B} = 0$

magnetic induction $\frac{\partial \vec{B}}{\partial t} - \frac{\partial}{\partial x}\hat{i} \times (\vec{u} \times \vec{B}) = 0$

Gauss law $\frac{\partial B_x}{\partial x} = 0$

pressure $\frac{\partial p}{\partial t} + u_x \frac{\partial p}{\partial x} + \gamma p \frac{\partial u_x}{\partial x} = 0$

1-demension ⇒ one variation only (here, x)

Linearization

initial condition

[illegible]

[illegible] = constant

Linearization

$$\frac{\partial \delta\rho}{\partial t} + \frac{\partial}{\partial x}\left(\rho_0 \delta v_x + v_{x0}\delta\rho\right) = 0 \qquad \delta\vec{u} \ll 1$$

more than second order terms dropped out

$$\rho_0 \frac{\partial \delta v_x}{\partial t} + \frac{\partial}{\partial x}\left(\rho_0 v_{x0}\delta v_x + \delta p + B_{y0}\delta B_y + B_{z0}\delta B_z\right) = 0$$

$$\rho_0 \frac{\partial \delta v_y}{\partial t} + \frac{\partial}{\partial x}\left(\rho_0 v_{x0}\delta v_y - B_{x0}\delta B_y\right) = 0$$

$$\rho_0 \frac{\partial \delta v_z}{\partial t} + \frac{\partial}{\partial x}\left(\rho_0 v_{x0}\delta v_z - B_{x0}\delta B_z\right) = 0$$

$$\delta\vec{u} \propto e^{i(\omega t - kx)}$$

$$\frac{\partial \delta B_y}{\partial t} + \frac{\partial}{\partial x}\left(B_{y0}\delta v_x - B_{x0}\delta v_y + v_{x0}\delta B_y\right) = 0$$

$$\frac{\partial \delta B_z}{\partial t} + \frac{\partial}{\partial x}\left(B_{z0}\delta v_x - B_{x0}\delta v_z + v_{x0}\delta B_z\right) = 0$$

$$\frac{\partial \delta p}{\partial t} + \frac{\partial}{\partial x}\left(\gamma p_0 \delta v_x + v_{x0}\delta p\right) = 0$$

Linearization

$$\omega \delta\rho - k\left(\rho_0 \delta v_x + v_{x0} \delta\rho\right) = 0$$

$$\omega \rho_0 \delta v_x - k\left(\rho_0 v_{x0} \delta v_x + \delta p + B_{y0} \delta B_y + B_{z0} \delta B_z\right) = 0$$

$$\omega \rho_0 \delta v_y - k\left(\rho_0 v_{x0} \delta v_y - B_{x0} \delta B_y\right) = 0$$

$$\omega \rho_0 \delta v_z - k\left(\rho_0 v_{x0} \delta v_z - B_{x0} \delta B_z\right) = 0$$

$$\omega \delta B_y - k\left(B_{y0} \delta v_x - B_{x0} \delta v_y + v_{x0} \delta B_y\right) = 0$$

$$\omega \delta B_z - k\left(B_{z0} \delta v_x - B_{x0} \delta v_z + v_{x0} \delta B_z\right) = 0$$

$$\omega \delta p - k\left(\gamma p_0 \delta v_x + v_{x0} \delta p\right) = 0$$

Eigen mode

$$a = -\frac{\omega}{k} = v_{x0} + \lambda$$

Eigen modes

$$
\begin{bmatrix}
\lambda & -\rho_0 & 0 & 0 & 0 & 0 & 0 \\
0 & \rho_0\lambda & 0 & 0 & -B_{y0} & -B_{z0} & -1 \\
0 & 0 & \rho_0\lambda & 0 & B_{x0} & 0 & 0 \\
0 & 0 & 0 & \rho_0\lambda & 0 & B_{x0} & 0 \\
0 & -B_{y0} & B_{x0} & 0 & \lambda & 0 & 0 \\
0 & -B_{z0} & 0 & B_{x0} & 0 & \lambda & 0 \\
0 & -\gamma p_0 & 0 & 0 & 0 & 0 & \lambda
\end{bmatrix}
\begin{bmatrix}
\delta\rho \\ \delta u_x \\ \delta u_y \\ \delta u_z \\ \delta B_y \\ \delta B_z \\ \delta p
\end{bmatrix} = 0
$$

$$c_{s0}^2 = \frac{\gamma p_0}{\rho_0}$$

$$b_{i0} = \frac{B_{i0}}{\sqrt{\rho_0}}$$

$$b_0^2 = b_{x0}^2 + b_{y0}^2 + b_{z0}^2$$

$$\det(|\ \ |) = \lambda\,(\lambda^2 - b_{x0}^2)\left[\lambda^2(\lambda^2 - b_0^2) - c_{s0}^2(\lambda^2 - b_{x0}^2)\right]$$

entropy Alfven compressible(fast, slow)

1. entropy $\lambda| = 0$

2. Alfven $\lambda^2 = b_{x0}^2$

3. compressible $\lambda^2(\lambda^2 - b_0^2) - c_{s0}^2(\lambda^2 - b_{x0}^2) = 0$

Wave analysis

1. entropy [illegible]

[illegible]

2. Alfven [illegible]

[illegible]

[illegible]

3. compressible [illegible]

[illegible]

[illegible]

Alfven wave test

2. +Alfven $\lambda = b_{x0}$

Initial condition for plus Alfven wave

Backgrounds

$$
\begin{aligned}
\rho_0 &= 1 \\
v_{x0} &= 0 \\
v_{y0} &= 0 \\
v_{z0} &= 0 \\
B_{x0} &= 1 \\
B_{y0} &= 0.5 \\
B_{z0} &= 0 \\
p_0 &= 1/\gamma
\end{aligned}
$$

Perturbations

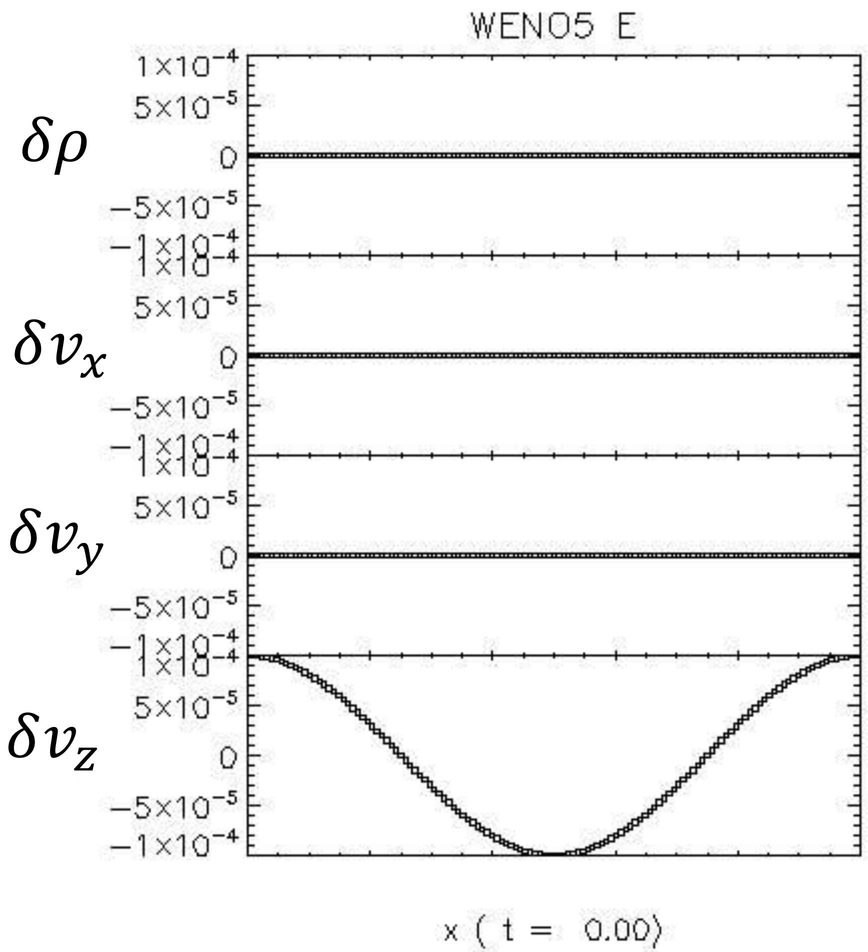
$$
\begin{aligned}
\delta\rho &= 0 \\
\delta v_x &= 0 \\
\delta v_y &= 0 \\
\delta v_z &= 10^{-4}\cos(kx) \\
\delta B_x &= 0 \\
\delta B_y &= 0 \\
\delta B_z &= -\sqrt{\rho_0}\,\delta v_z \\
\delta p &= 0
\end{aligned}
$$

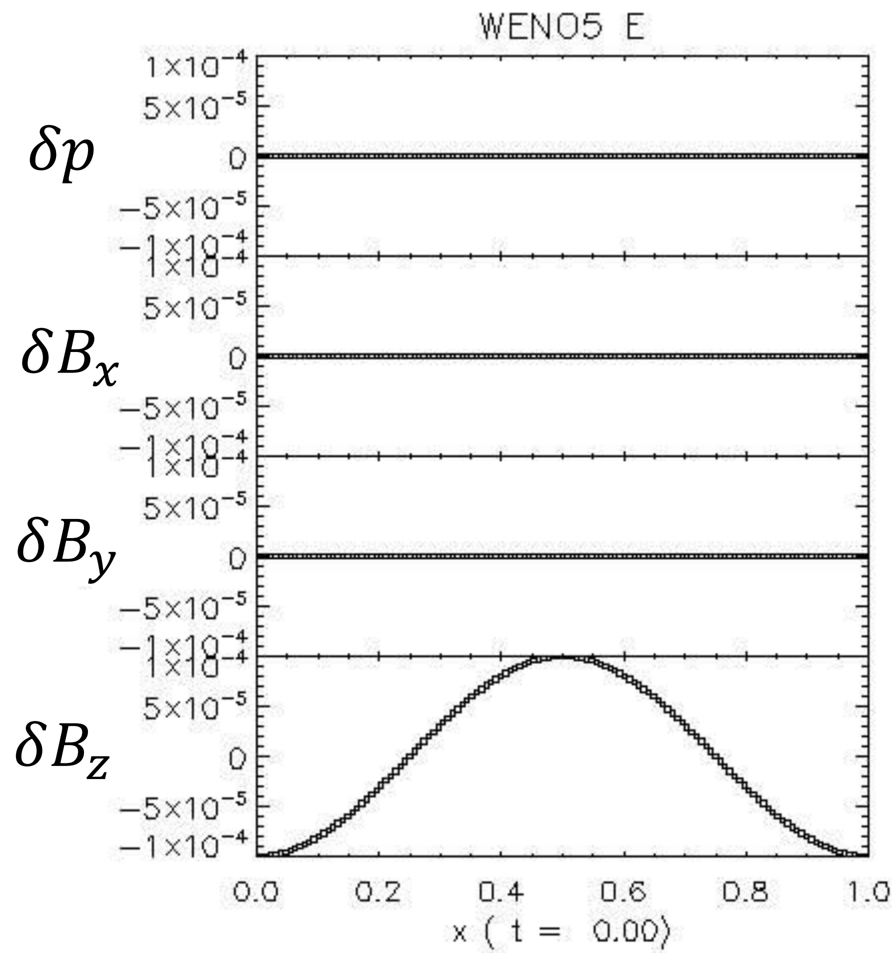
Total

$$
\begin{aligned}
\rho &= \rho_0 + \delta\rho \\
v_x &= v_{x0} + \delta v_x \\
v_y &= v_{y0} + \delta v_y \\
v_z &= v_{z0} + \delta v_z \\
B_x &= B_{x0} \\
B_y &= B_{y0} + \delta B_y \\
B_z &= B_{z0} + \delta \mathrm{B}_z \\
p &= p_0 + \delta p
\end{aligned}
$$

Alfven wave test ⇨ 1. Code verification

Test Code
- MHD 5th order WENO
- periodic boundary condition
- resolution = 128 , tend = 1.0





Alfven wave test ⇨ 1. Code verification

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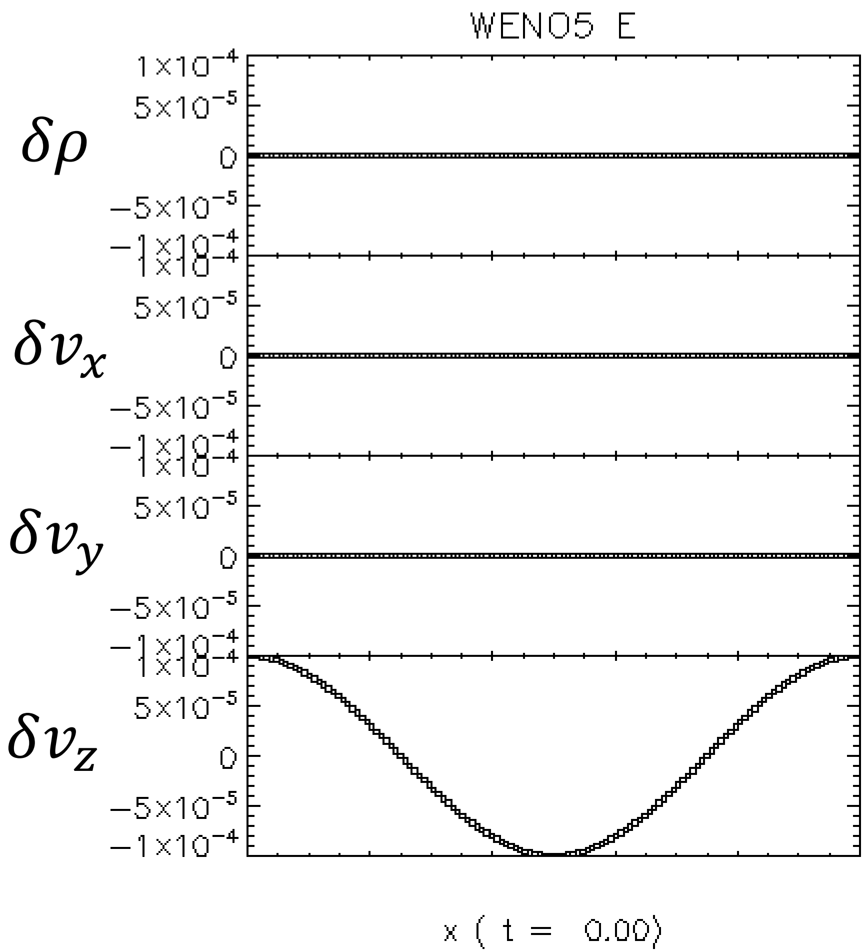
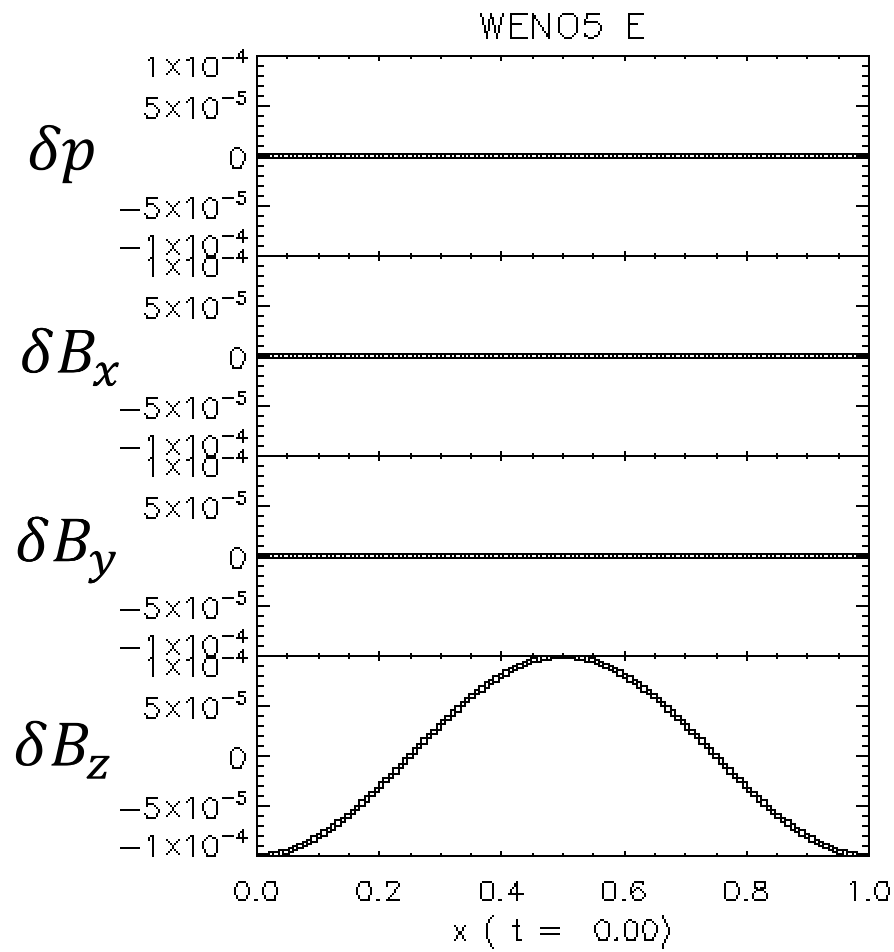
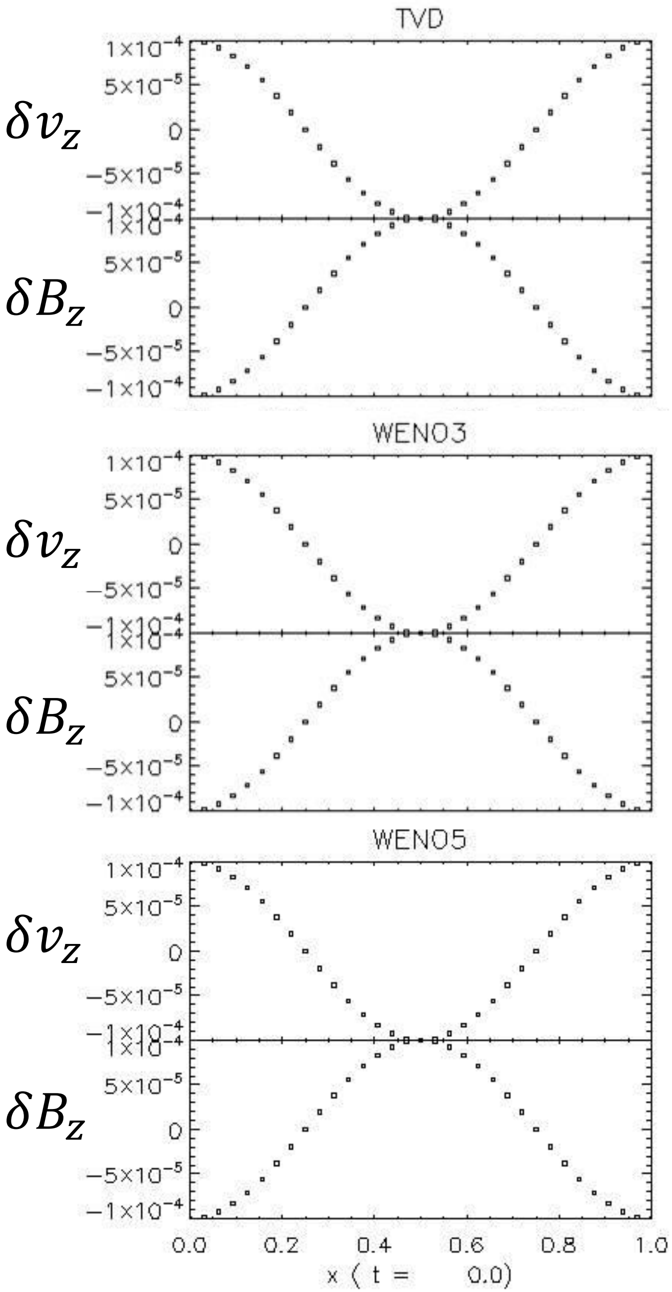




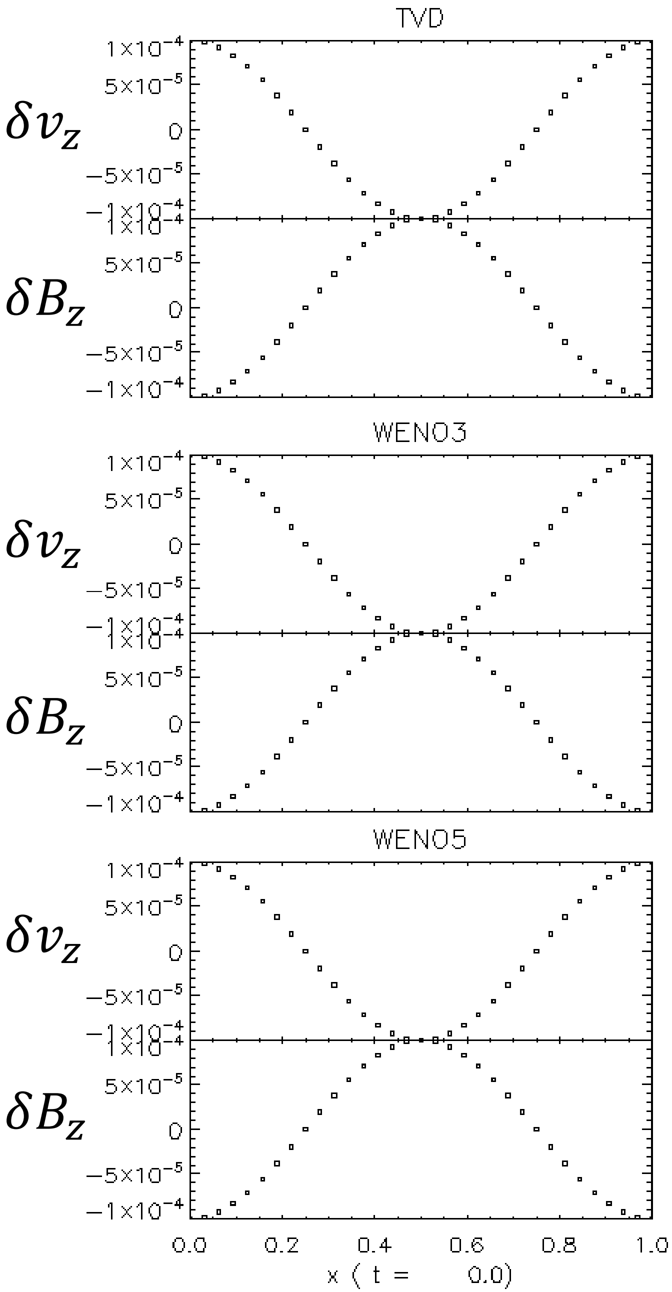
Alfven wave test ⇨ 2. Order Check



Test Code

- 2nd order TVD
- 3rd order WENO3
- 5th order WENO5
- resolution =32 tend =100

Alfven wave test ⇨ 2. Order Check



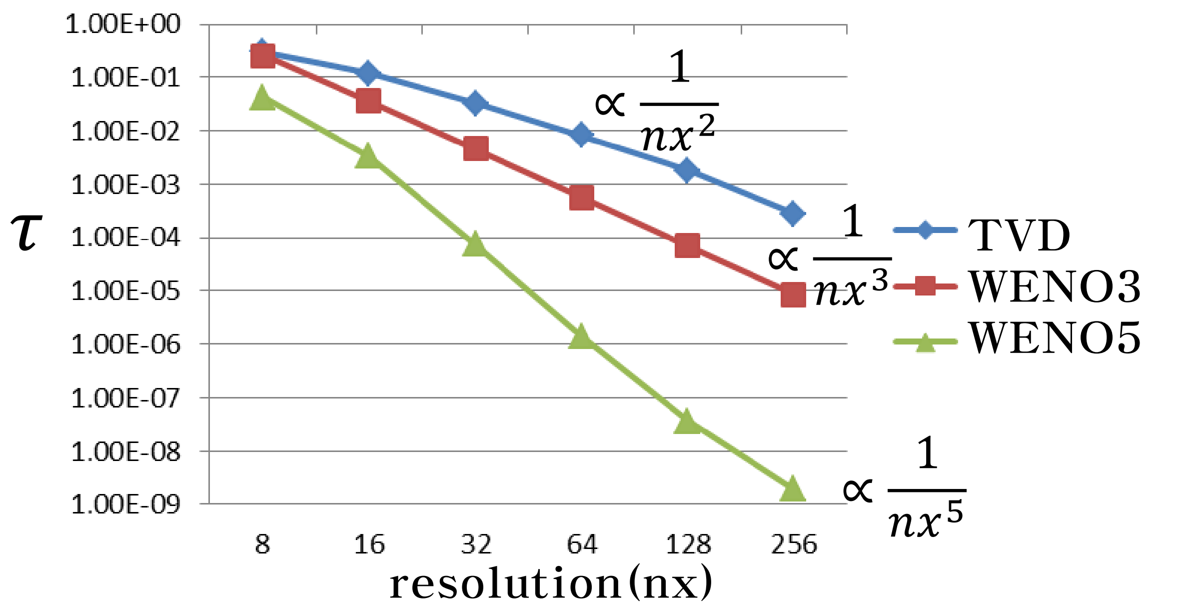
Test Code

- 2nd order TVD
- 3rd order WENO3
- 5th order WENO5
- resolution =32 tend =100

Numerical dissipation

$$\frac{\delta v_z(tend)}{\delta v_z(t=0)} = e^{-tend \cdot \tau}$$

$$\tau = -\frac{1}{tend} ln\left(\frac{\delta v_z(tend)}{\delta v\_z(t=0)}\right)$$ decay rate



2. Cylindrical Coordinate

MHD equations

Primitive forms

density $$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{v} \right) = 0$$

velocity $$\frac{\partial \vec{v}}{\partial t} + \left(\vec{v} \cdot \vec{\nabla} \right) \vec{v} + \frac{1}{\rho} \vec{\nabla} p - \frac{1}{\rho} \left(\vec{\nabla} \times \vec{B} \right) \times \vec{B} = 0$$

magnetic induction $$\frac{\partial \vec{B}}{\partial t} - \vec{\nabla} \times \left(\vec{v} \times \vec{B} \right) = 0$$

Gauss law $$\vec{\nabla} \cdot \vec{B} = 0$$

γ adiabatic index

pressure $$\frac{\partial p}{\partial t} + \left(\vec{v} \cdot \vec{\nabla} \right) p + \gamma p \left(\vec{\nabla} \cdot \vec{v} \right) = 0$$

2. Cylindrical Coordinate

MHD equations

$$\frac{\partial \delta\rho}{\partial t} + \rho_0 \frac{\partial \delta v_r}{\partial r} + \frac{\rho_0}{r}\delta v_r = 0$$

$$\rho_0 \frac{\partial \delta v_r}{\partial t} + \frac{\partial \delta p}{\partial r} + B_{\theta 0}(r)\frac{\partial \delta B_\theta}{\partial r} + \frac{1}{r}\delta B_\theta = 0$$

$$\frac{\partial \delta v_\theta}{\partial t} = 0$$

$$\frac{\partial \delta v_z}{\partial t} = 0$$

$$\frac{\partial \delta B_\theta}{\partial t} + B_{\theta 0}(r)\frac{\partial \delta v_r}{\partial r} - \frac{B_\theta}{r}\delta v_r = 0$$

$$\frac{\partial \delta B_z}{\partial t} = 0$$

$$\frac{\partial \delta p}{\partial t} + \gamma p_0 \frac{\partial \delta v_r}{\partial r} + \frac{\gamma}{r} p_0 \delta v_r = 0$$

Background equilibrium

$$B_{\theta 0}(r) = B_0\left(\frac{R_0}{r}\right)$$

1D (r-dependence only)

$$\vec{\nabla} f = \frac{\partial f}{\partial r}\hat{r} \qquad \vec{\nabla}\cdot\vec{F} = \frac{1}{r}\frac{\partial (rF_r)}{\partial r}\hat{r}$$

$$\vec{\nabla}\times\vec{F} = -\frac{\partial F_z}{\partial r}\hat{\theta} + \frac{1}{r}\frac{\partial (rF_\theta)}{\partial r}\hat{z}$$

Fast wave assumption

$$\rho = \rho_0 + \delta\rho$$
$$v_r = \delta v_r$$
$$v_\theta = \delta v_\theta$$
$$v_z = \delta v_z$$
$$B_r = 0$$
$$B_\theta = B_{\theta 0}(r) + \delta B_\theta$$
$$B_z = \delta \mathrm{B}_z$$
$$p = p_0 + \delta p$$

Wave analysis

$$\frac{\partial}{\partial t}\left(\rho_0 \frac{\partial \delta v_r}{\partial t} + \frac{\partial \delta p}{\partial r} + B_{\theta 0}(r)\frac{\partial \delta B_\theta}{\partial r} + \frac{1}{r}\delta B_\theta\right) = 0$$

$$\Rightarrow \rho_0 \frac{\partial^2 \delta v_r}{\partial t^2} + \frac{\partial}{\partial r}\left(\frac{\partial \delta p}{\partial t}\right) + B_{\theta 0}(r)\frac{\partial}{\partial r}\left(\frac{\partial \delta B_\theta}{\partial t}\right) + \frac{1}{r}\frac{\partial \delta B_\theta}{\partial t} = 0$$

$$\frac{\partial \delta p}{\partial t} + \gamma p_0 \frac{\partial \delta v_r}{\partial r} + \frac{\gamma}{r} p_0 \delta v_r = 0 \qquad \frac{\partial \delta B_\theta}{\partial t} + B_{\theta 0}(r)\frac{\partial \delta v_r}{\partial r} - \frac{B_\theta}{r}\delta v_r = 0$$

$$\delta \vec{u}(t) \propto e^{iwt}$$

$$c_s = \sqrt{\gamma \frac{p_0}{\rho_0}} \quad c_A = \sqrt{\frac{B_0^2}{\rho_0}} \quad R = \frac{r}{R_0} \quad W = wR_0$$

$$\left(\frac{1}{R^2} + \frac{c_s^2}{c_A^2}\right)\frac{\partial^2 \delta v_r}{\partial r^2} - \frac{1}{R}\left(\frac{1}{R^2} - \frac{c_s^2}{c_A^2}\right)\frac{\partial \delta v_r}{\partial r} + \frac{1}{R^2}\left(\frac{1}{R^2} - \frac{c_s^2}{c_A^2} + \frac{W^2R^2}{c_A^2}\right)\delta v_r = 0$$

When $\frac{c_s^2}{c_A^2} = 0.01$, numerically find $\frac{W^2 R_0^2}{c_A^2}$ $\Rightarrow$ $= 44.744$

At $R = 1(= R_0)$, $\delta v_r = 0, \frac{\partial \delta v_r}{\partial r} = 1$

At $R = 1.7$, $\delta v_r = 0, \frac{\partial \delta v_r}{\partial r} = 1$

$R_0 = 1$, $c_A = 10 \rightarrow \omega = 66.89$

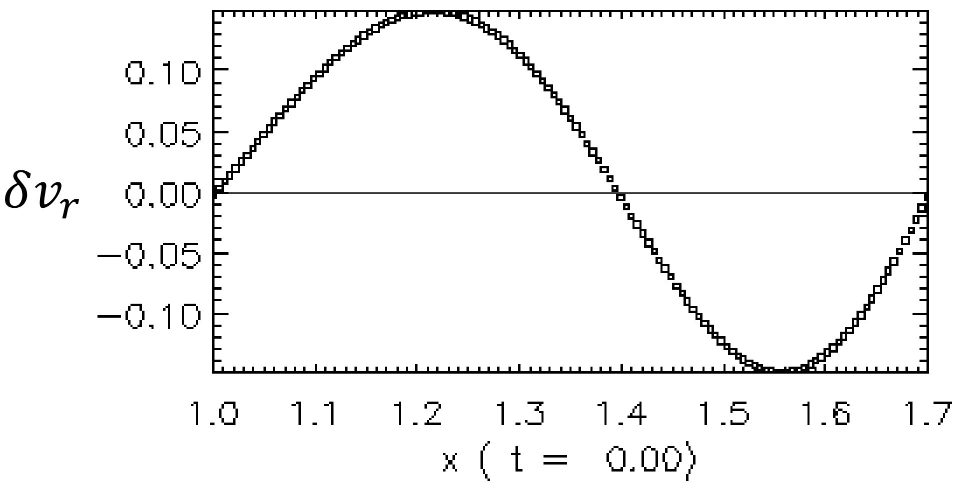
Period $= \frac{2\pi}{\omega} = 0.094$

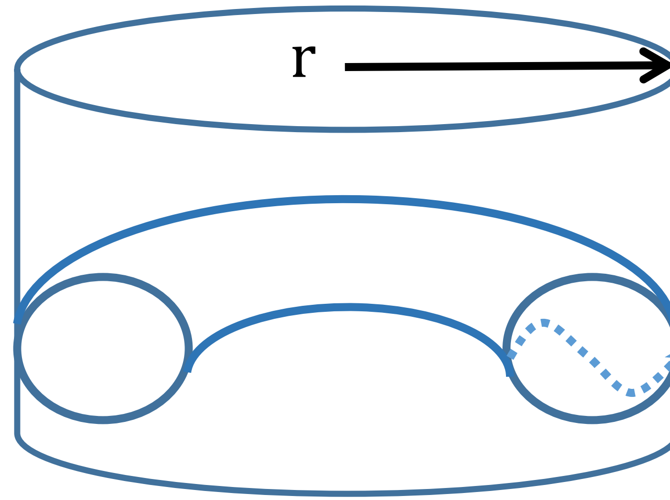
Fast wave test

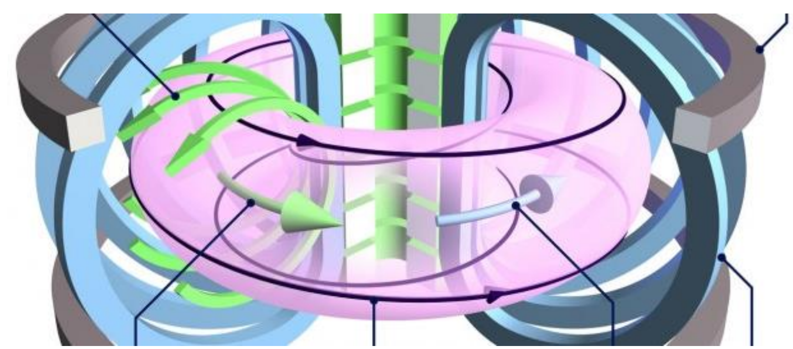
$R_0 = 1,\ c_s = 1,\ c_A = 10,\ W = 6.689$

$$\left(\frac{1}{R^2}+\frac{c_s^2}{c_A^2}\right)\frac{\partial^2 \delta v_r}{\partial r^2}-\frac{1}{R}\left(\frac{1}{R^2}-\frac{c_s^2}{c_A^2}\right)\frac{\partial \delta v_r}{\partial r}+\frac{1}{R^2}\left(\frac{1}{R^2}-\frac{c_s^2}{c_A^2}+\frac{W^2R^2}{c_A^2}\right)\delta v_r=0$$

initial condition for δv_r





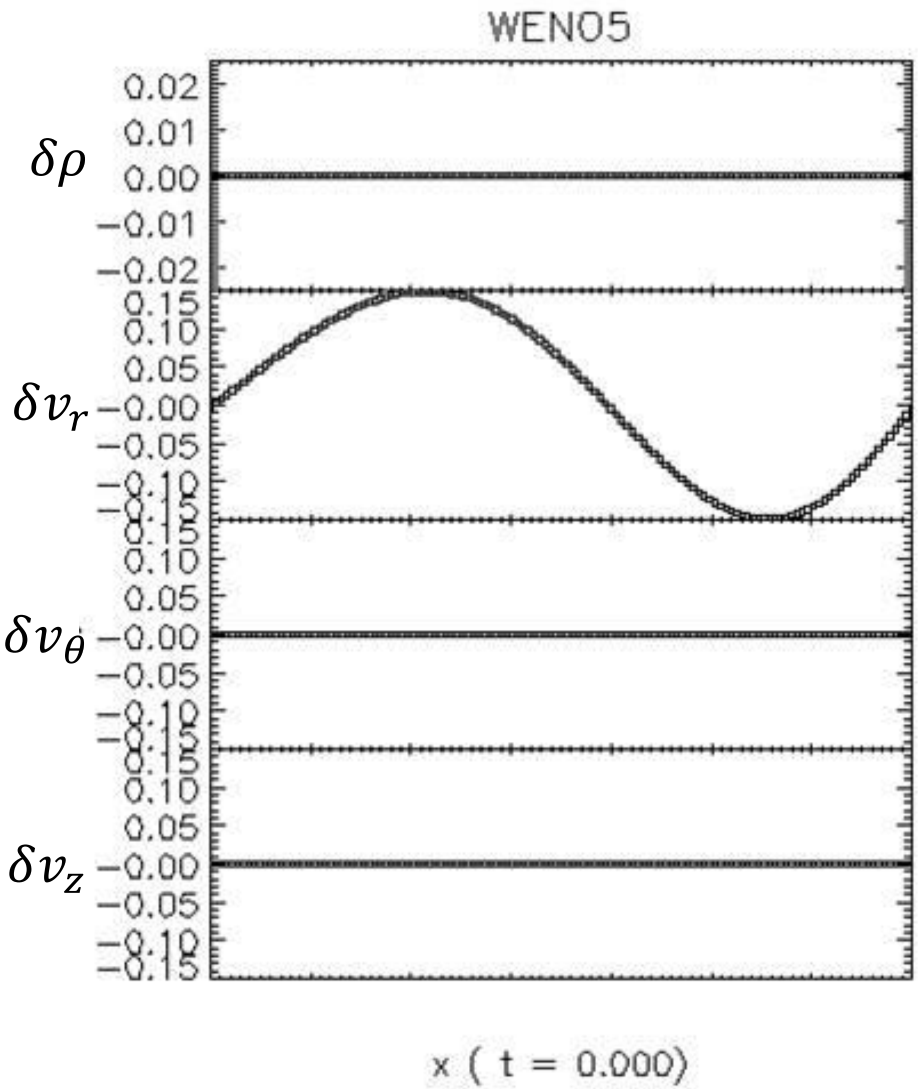


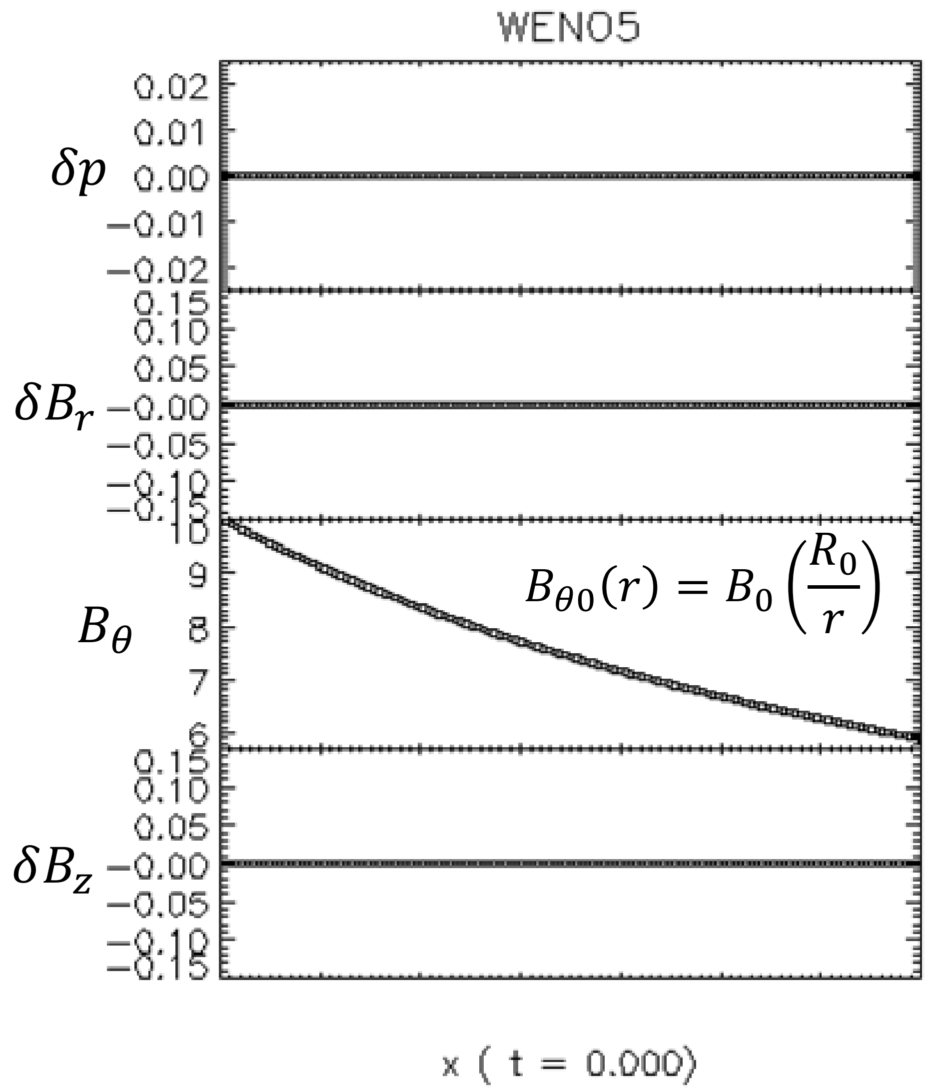
-Standing Wave
-Reflecting boundary
-Can be applied to TOKAMAK

Fast wave test

$\rho_0 = 1, B_0 = 10, p_0 = 1/\gamma$
$v_{r0} = v_{\theta 0} = v_{z0} = B_{r0} = B_{z0} = 0$

$c_s^2 = \gamma \frac{p_0}{\rho_0} = 1 \quad c_A^2 = \frac{B_0^2}{\rho_0} = 100$





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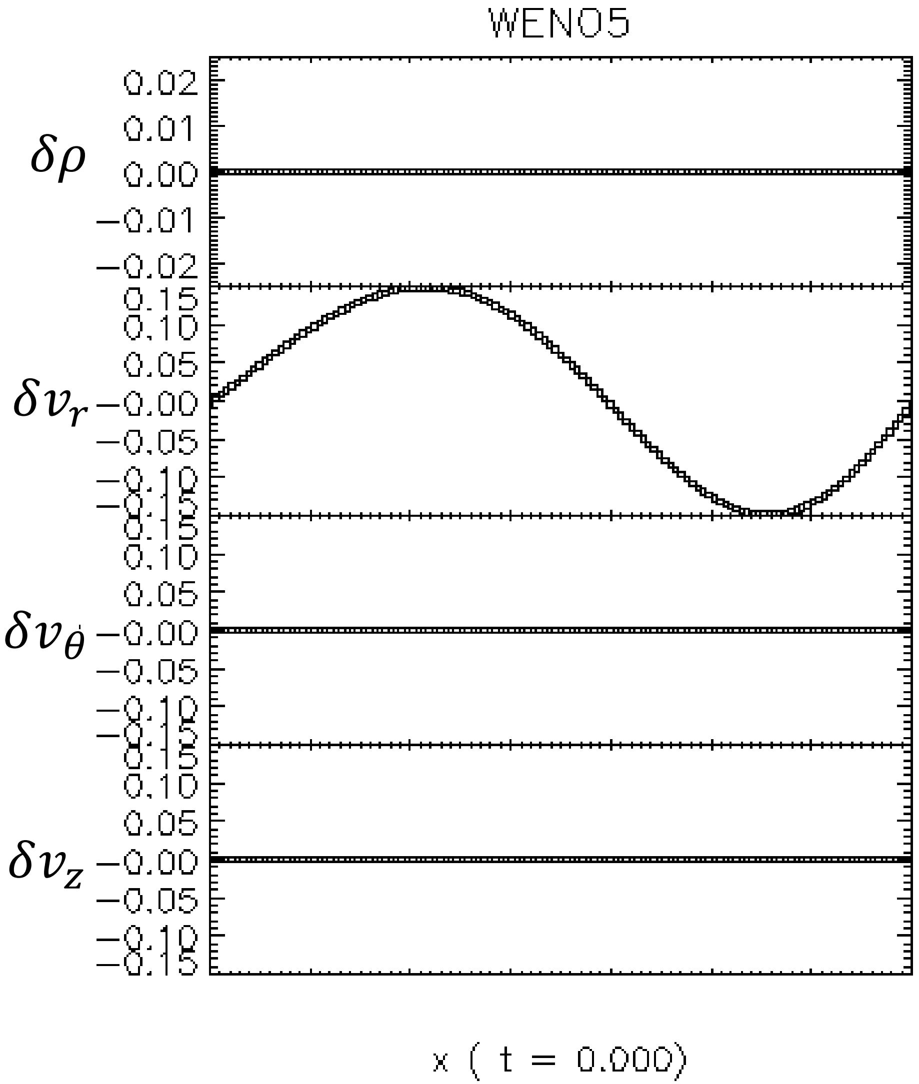

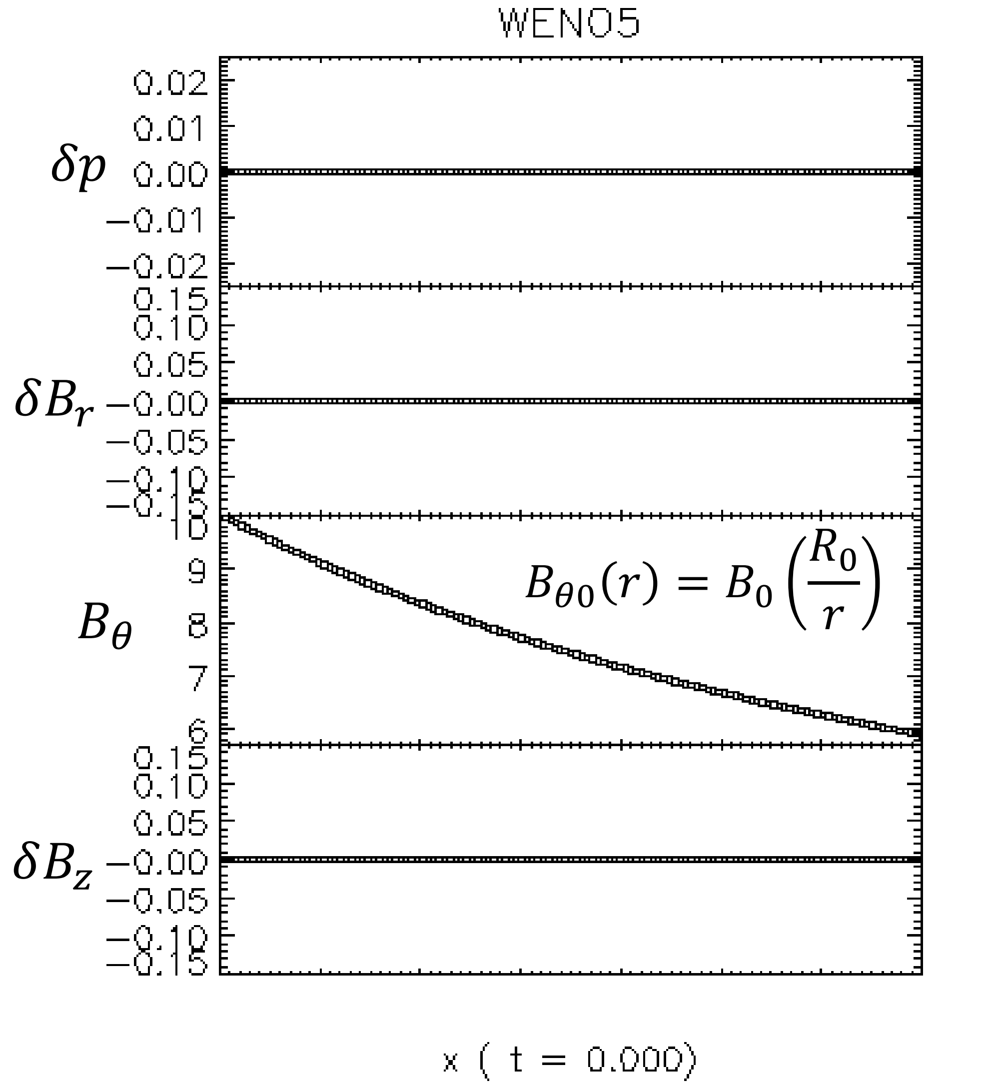

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