KNAG (2016/05/27)

Cosmic Microwave Background Temperature Anisotropy from the Kinetic Sunyaev-Zel'dovich Effect

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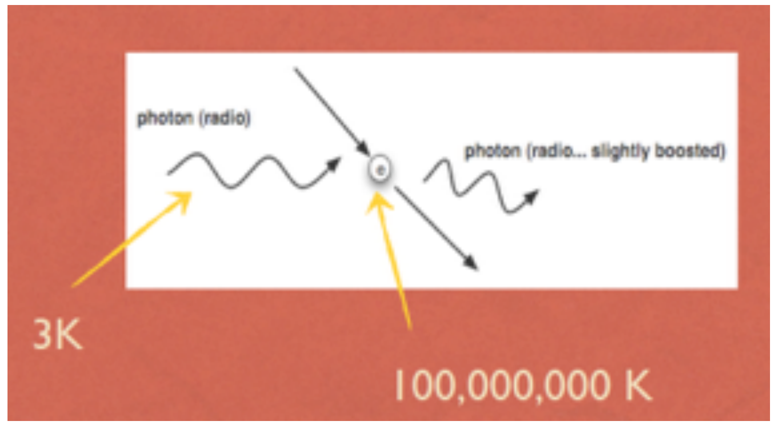
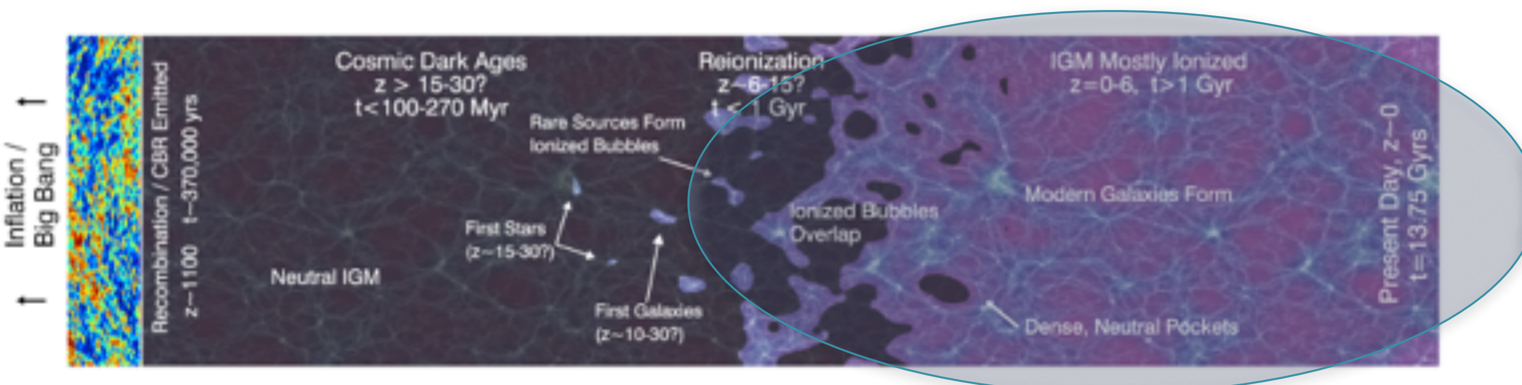
Intro) Sunyaev-Zel'dovich (SZ) Effect

Change of temperature due to
interaction between CMB photons and free electrons…



… produced by astrophysical sources in the post-Recombination era.

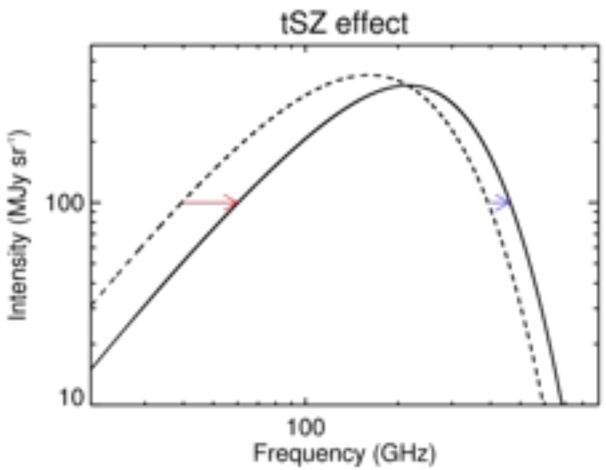


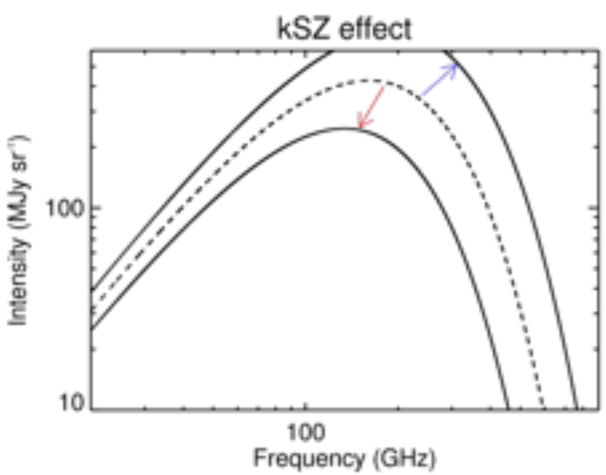
(Robertson et al. 2010)

Intro) SZ Effect

- Thermal SZ Effect (tSZ)
  Low energy photons up-scattered to have a higher frequency.

- Kinetic SZ Effect (kSZ)
  Doppler-shift by the bulk motion of the gas.

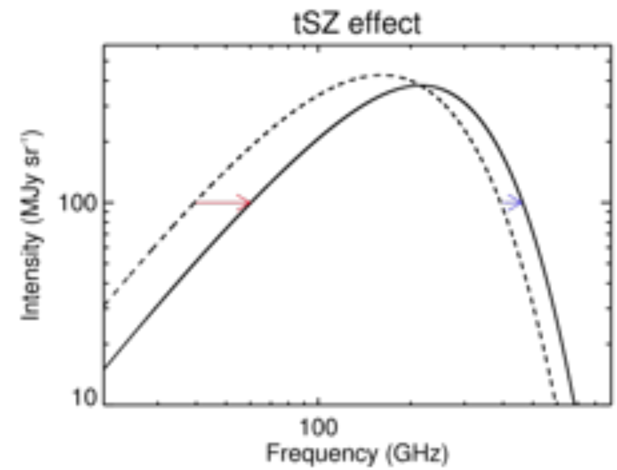




Intro) SZ Effect

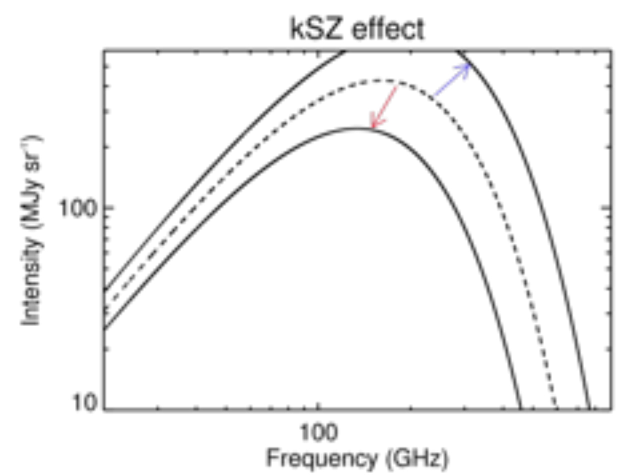
- Thermal SZ Effect (tSZ)

$$\frac{\Delta T}{T}(\hat{\gamma}) \propto \int d\tau \frac{kT}{m_e c^2}$$

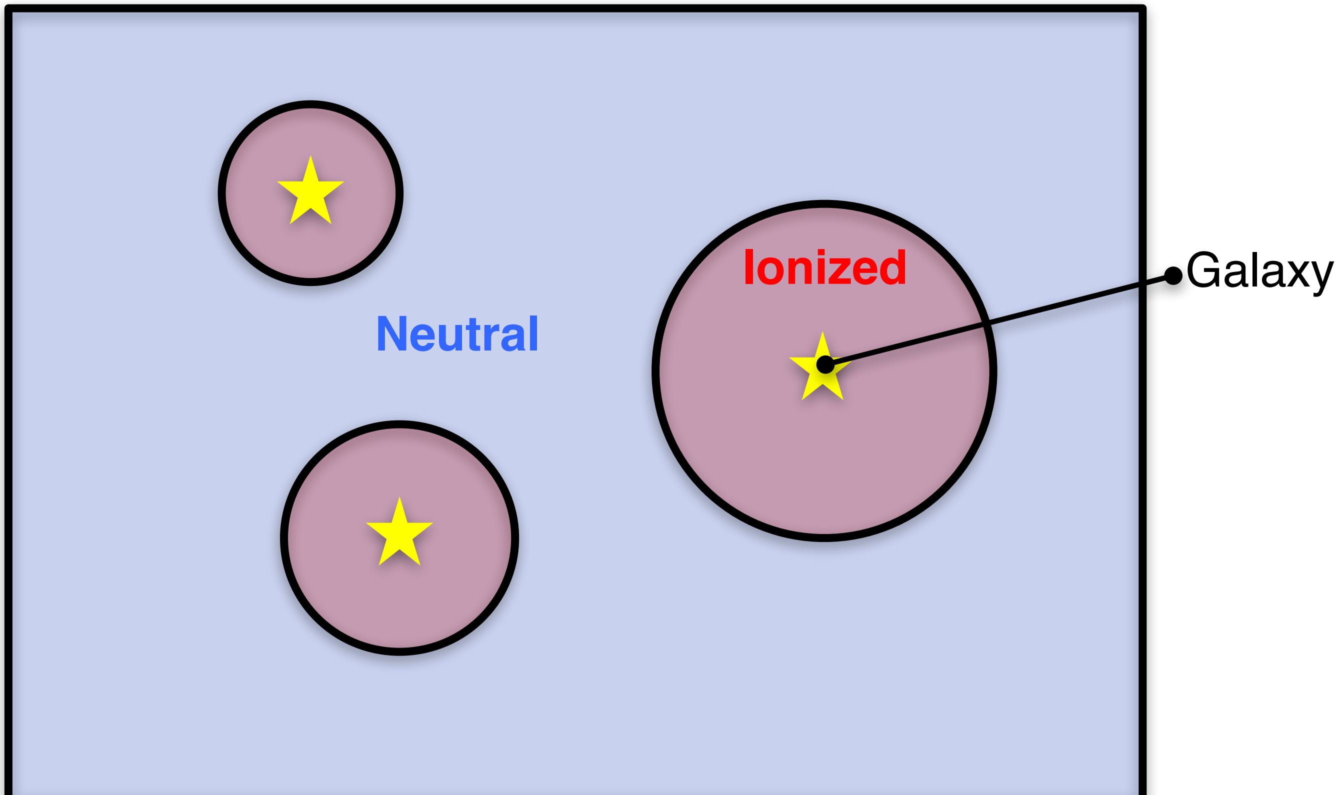


- Kinetic SZ Effect (kSZ)

$$\frac{\Delta T}{T}(\hat{\gamma}) = \int d\tau \hat{\gamma} \cdot \frac{\mathbf{v}}{c}$$

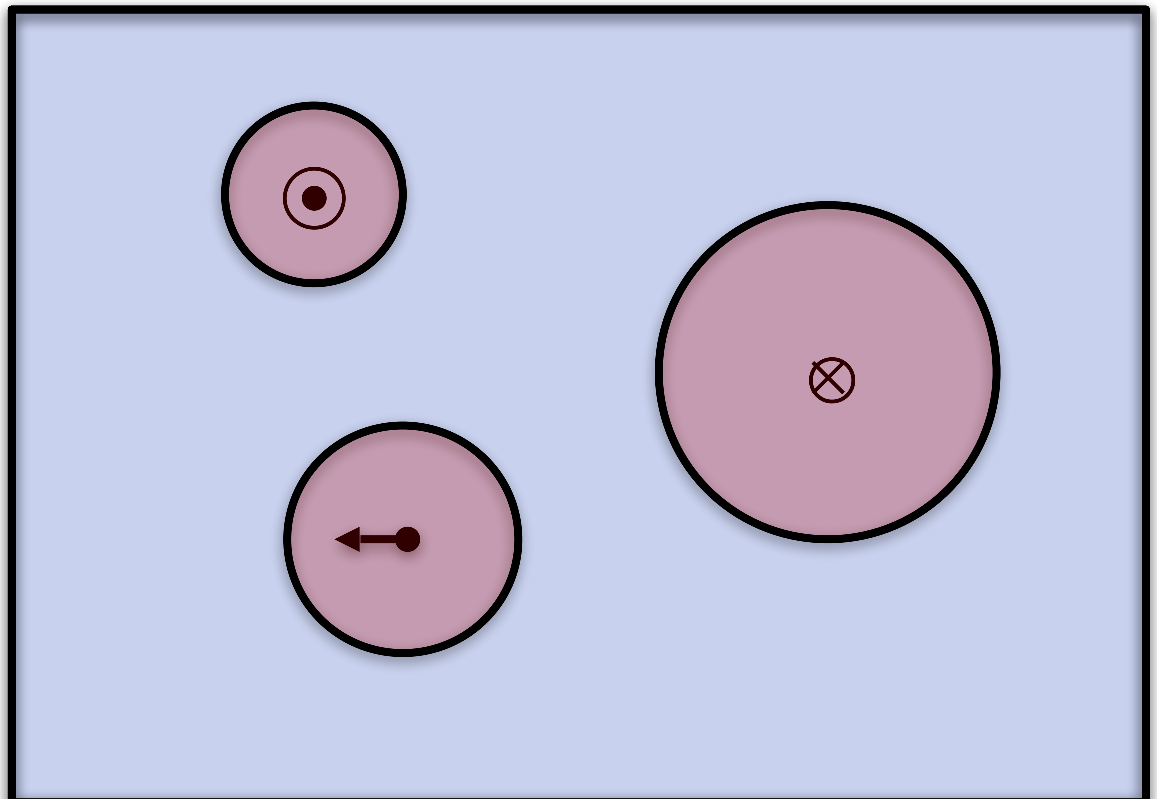


Ionized IGM around Galaxies



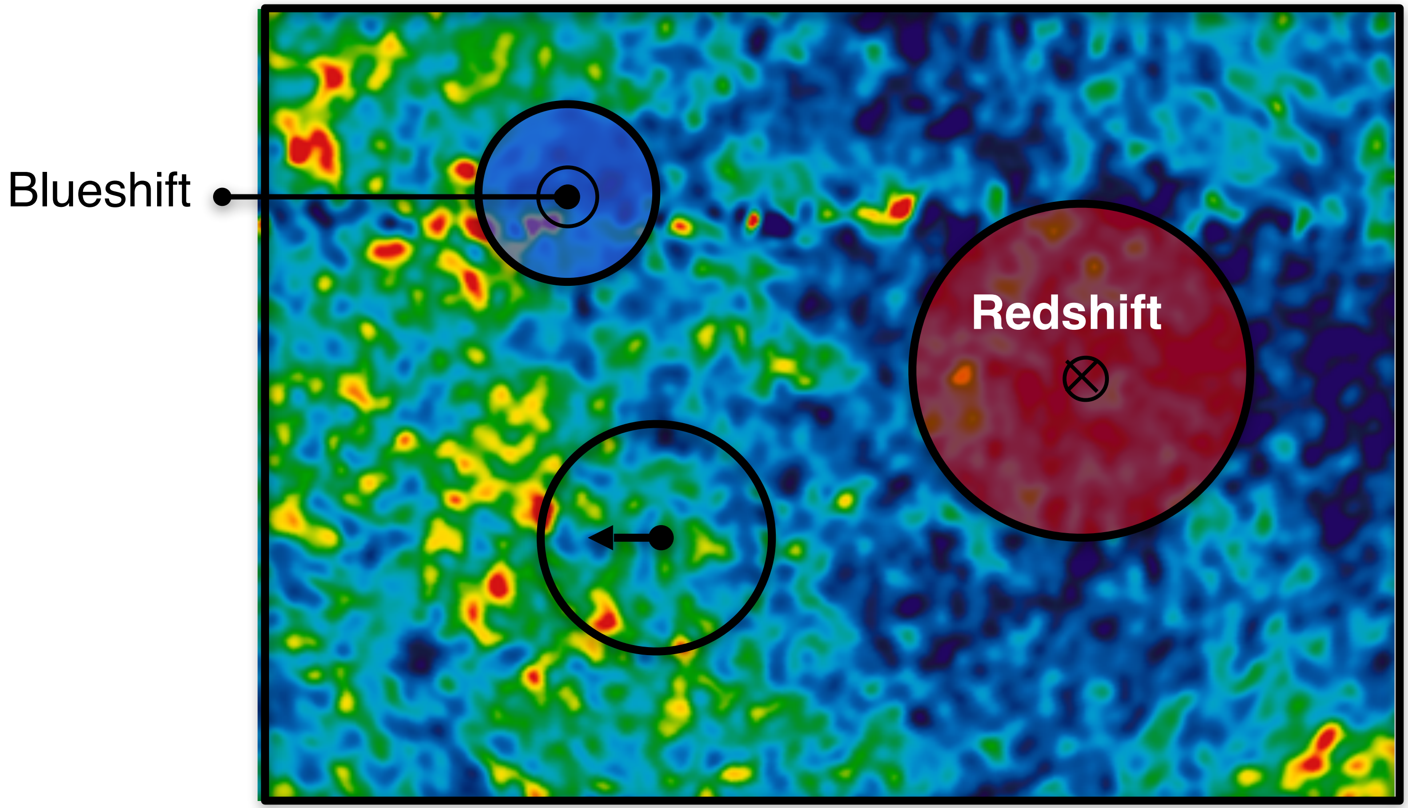
When the IGM has ionized parts, …

Ionized IGM with Peculiar Velocity



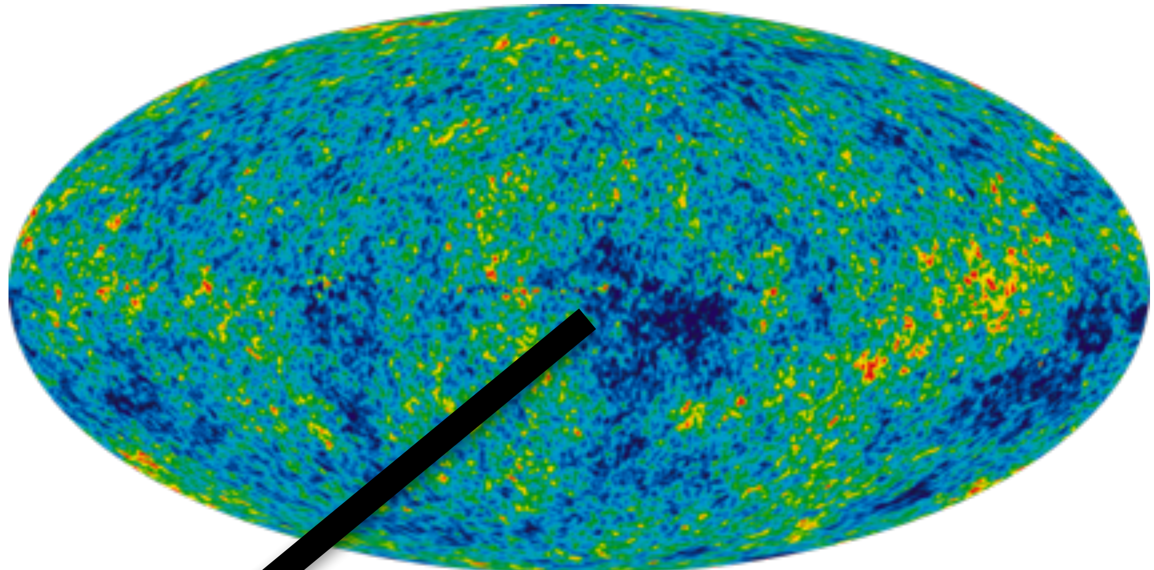
… each ionized patch can have different line-of-sight peculiar velocity.

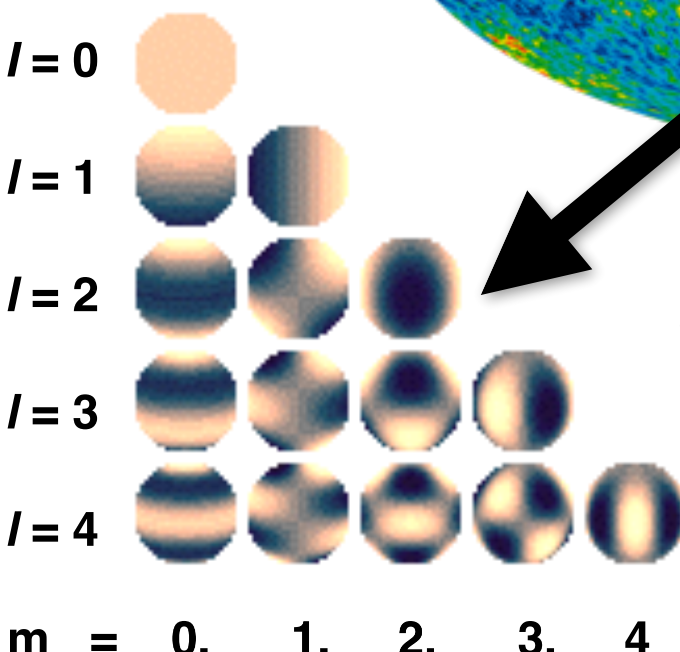
Kinetic SZ Effect



The kSZ effect would red/blueshift the CMB according to the line-of-sight velocity of the ionized IGM patches

CMB Power Spectrum



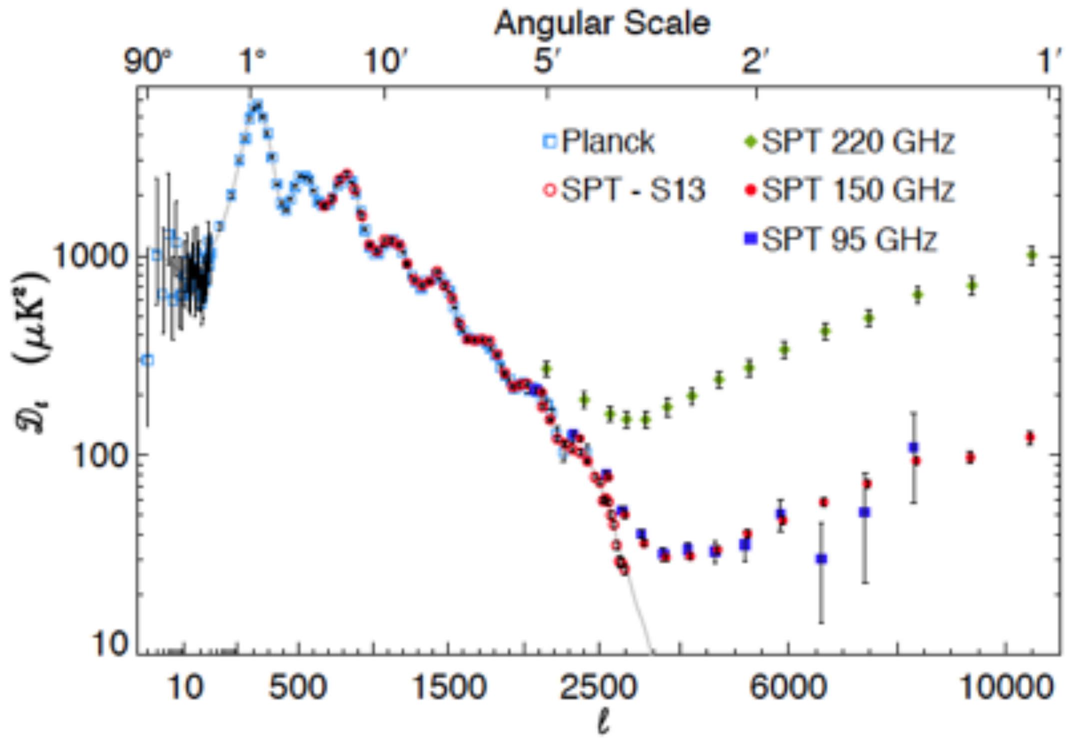


CMB temperature anisotropy can be decomposed into spherical harmonics.

$$a_{\ell m} \equiv \int d^2\hat{\gamma}\, Y_\ell^{m*}(\gamma)\frac{\Delta T}{T}(\hat{\gamma})$$

$$C_\ell = (2\ell+1)^{-1}\sum_{m=-\ell}^{\ell}|a_{\ell m}|^2$$

CMB Power Spectrum from the South-Pole Telescope

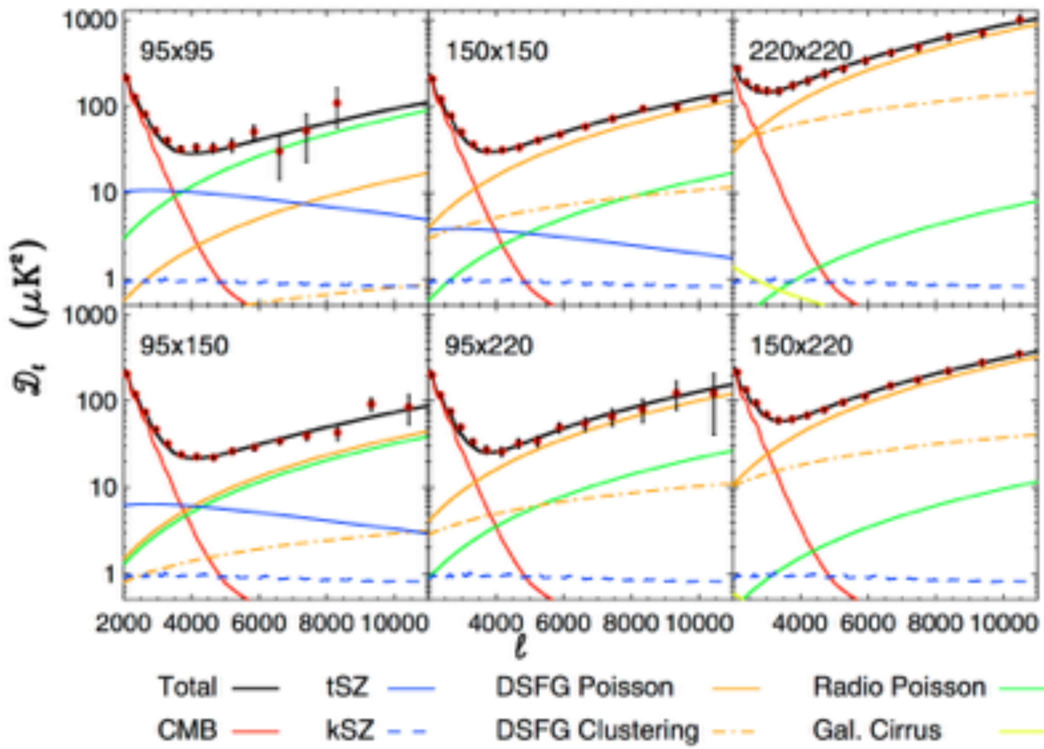


$$D_\ell \equiv (2\pi)^{-1}\ell(\ell+1)C_\ell$$

(George et al. 2015)

At $l > 2000$ or at angular scales below 5 arcmin, contribution from astrophysical sources dominate the signal.

kSZ Component in the CMB Power Spectrum



$$D_{\ell=3000}^{\mathrm{kSZ}} = 2.9 \pm 1.3\mu K^2$$ (George et al. 2015)

Using multi-frequency data, signals can be decomposed. For the kSZ effect, we only have rather a loose constraint at (*l* = 3000, 3 arcmin) at this point.

kSZ power spectrum

$$\frac{\Delta T}{T}(\hat{\gamma}) = \int d\tau \hat{\gamma} \cdot \frac{\mathbf{v}}{c}$$

$$\mathbf{q} \equiv X(1+\delta)\mathbf{v}$$

X: Ionized fraction

$$\frac{\Delta T}{T}(\hat{\gamma}) = \left(\frac{\bar{n}_{H,0}\sigma_T}{c}\right) \int \frac{ds}{a^2} (\mathbf{q} \cdot \hat{\gamma})$$

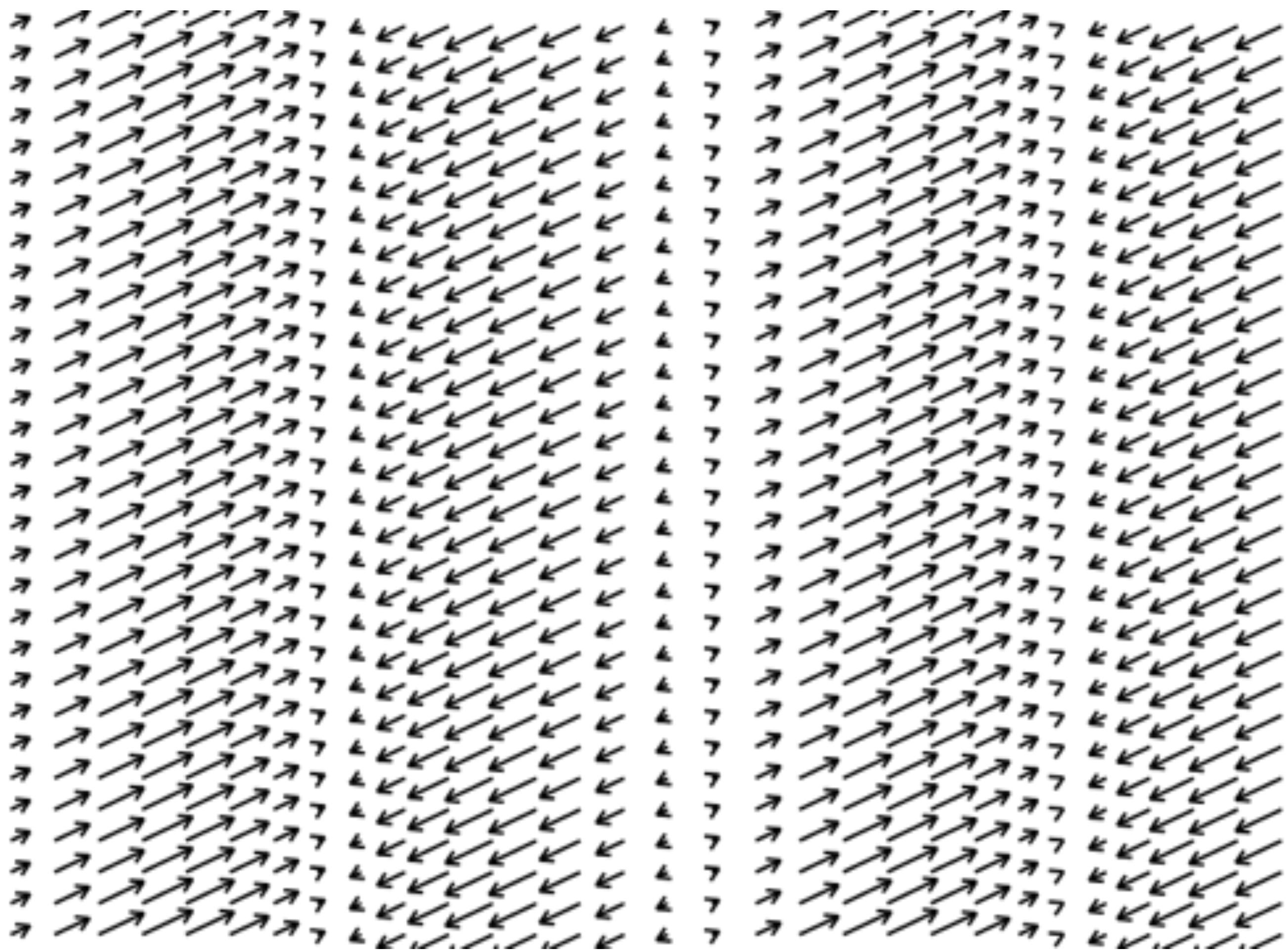
$$C_l = \left(\frac{\bar{n}_{H,0}\sigma_T}{c}\right)^2 \int \frac{ds}{s^2 a^4} \frac{P_{q_\perp}(k = l/s, z)}{2}$$

(See Park et al. 2013 for the derivation)

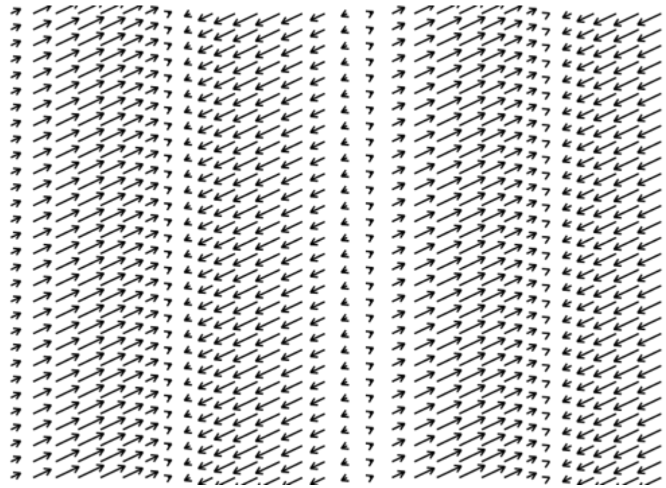
kSZ power spectrum is given by the transverse component of ionized momentum field.

What is the “Transverse” mode?

Here is a plane wave of a vector field.



The plane wave can be decomposed into…



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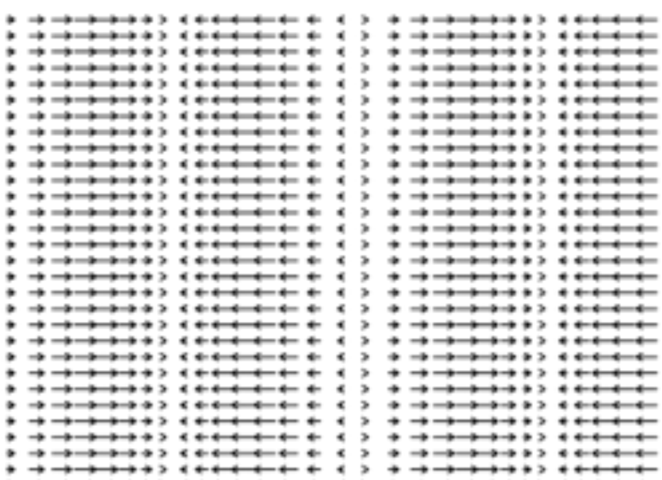
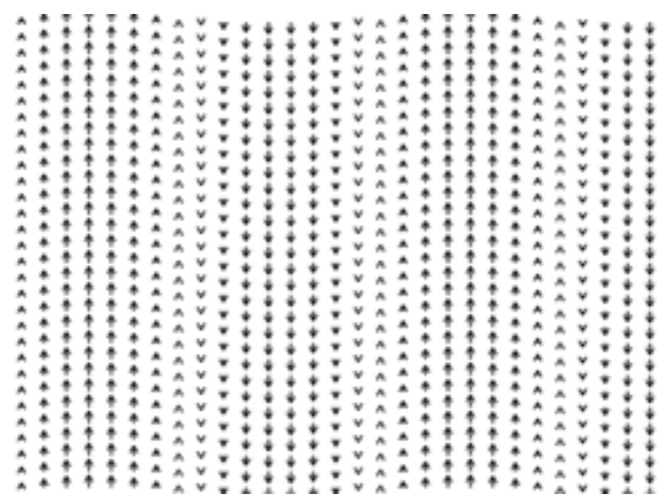
… the longitudinal mode and the transverse mode.



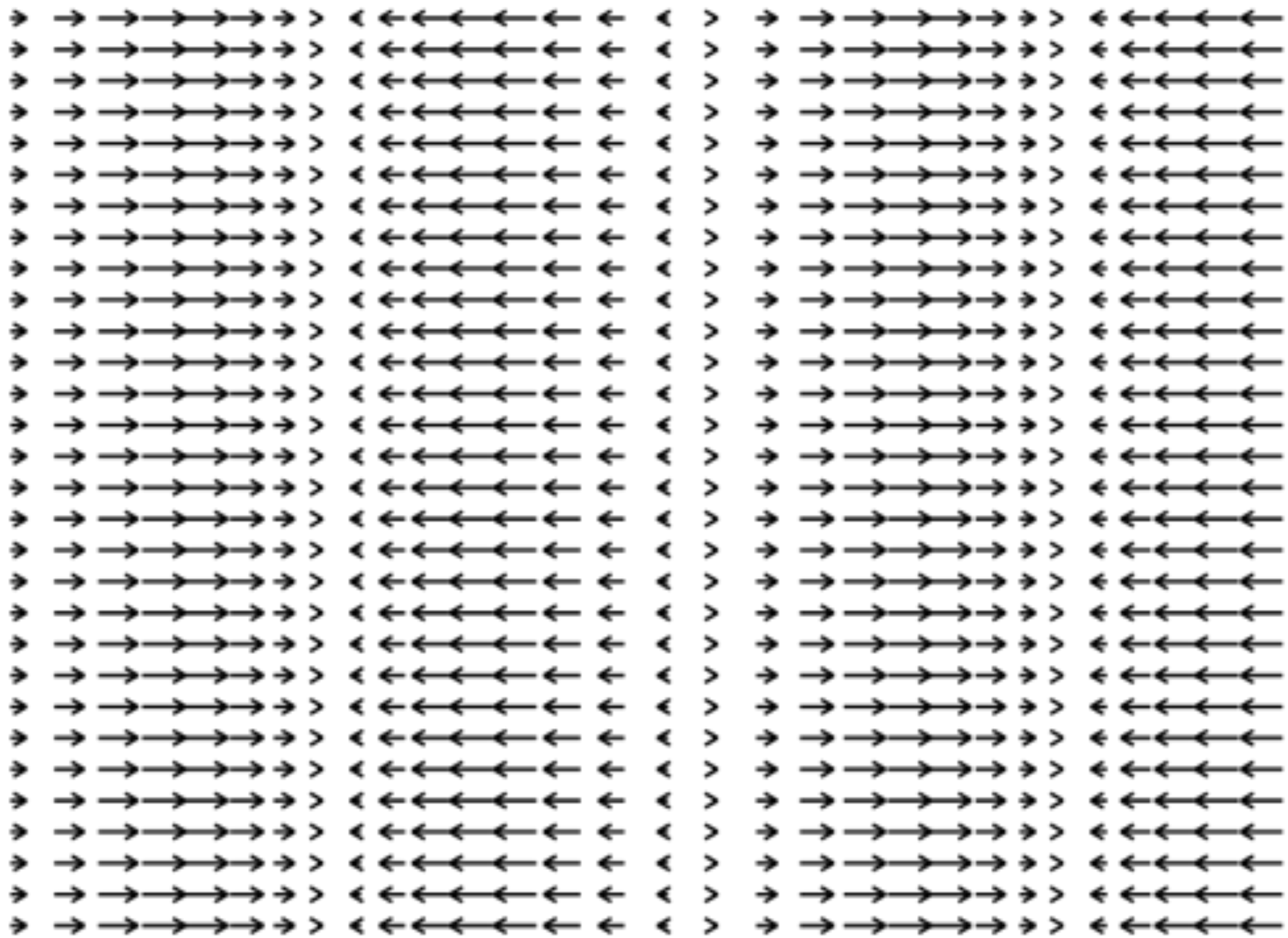
$$\tilde{\mathbf{q}}_{||} \equiv \hat{\mathbf{k}} \cdot \tilde{\mathbf{q}}$$

+



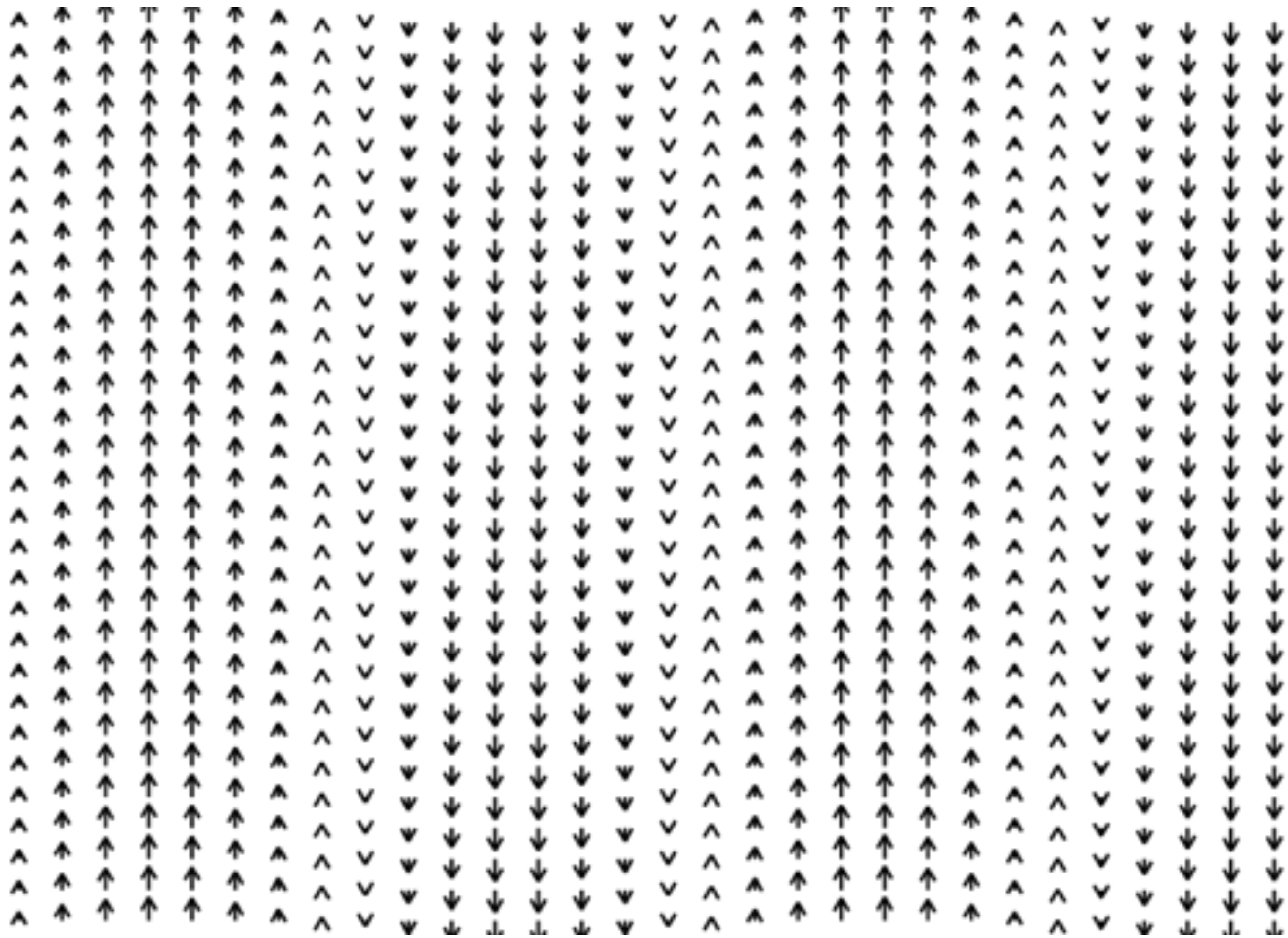
$$\tilde{\mathbf{q}}_{\perp} \equiv \tilde{\mathbf{q}} - \hat{\mathbf{k}} \cdot \tilde{\mathbf{q}}$$

Longitudinal Mode



No kSZ contribution from the longitudinal mode!

Transverse Mode



kSZ contribution when looking ↑

kSZ power spectrum

$$\frac{\Delta T}{T}(\hat{\gamma}) = \int d\tau \hat{\gamma} \cdot \frac{\mathbf{v}}{c}$$

$$\mathbf{q} \equiv X(1+\delta)\mathbf{v}$$

X: Ionized fraction

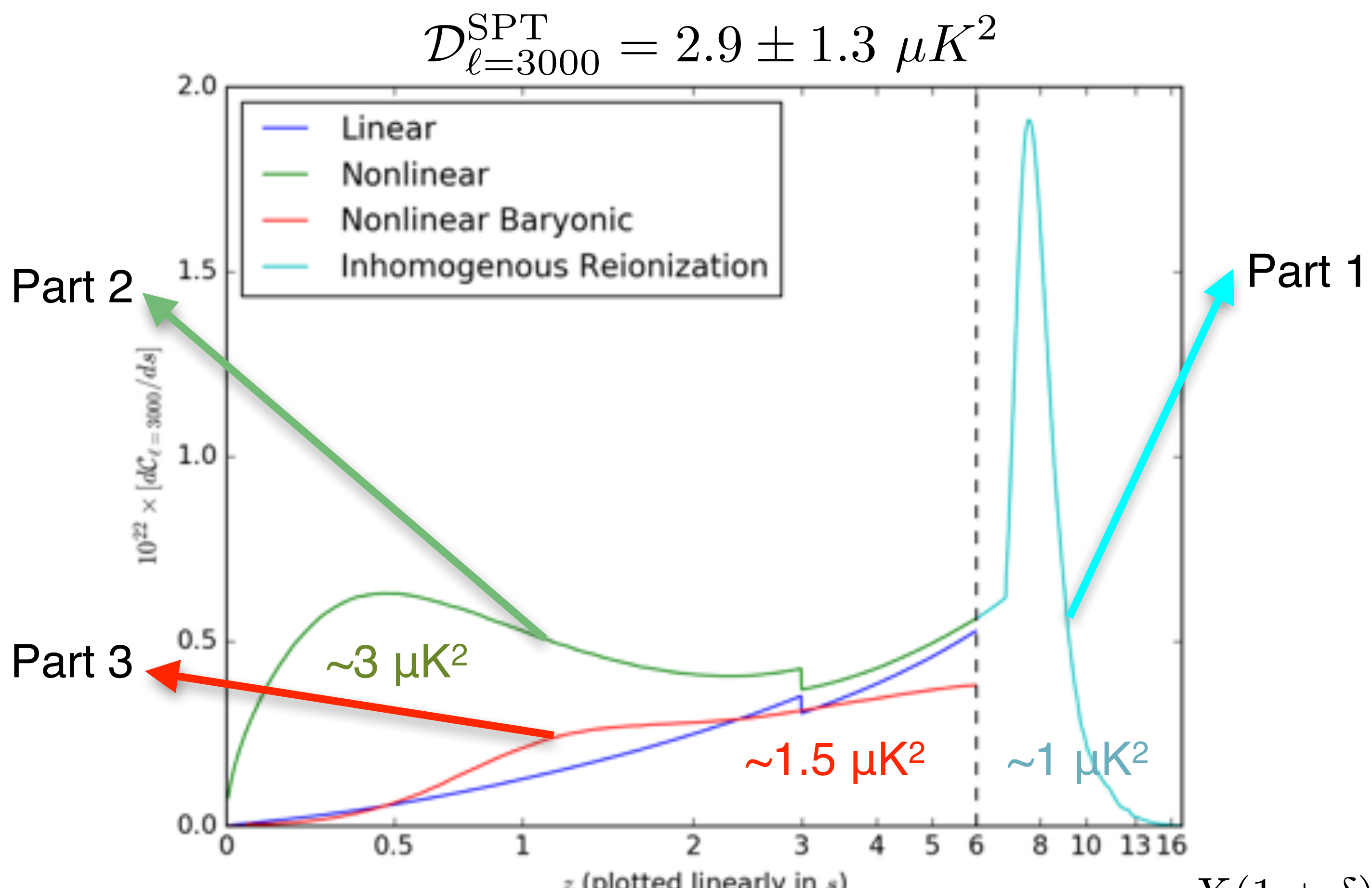
$$\frac{\Delta T}{T}(\hat{\gamma}) = \left(\frac{\bar{n}_{H,0}\sigma_T}{c}\right) \int \frac{ds}{a^2} (\mathbf{q} \cdot \hat{\gamma})$$

$$C_l = \left(\frac{\bar{n}_{H,0}\sigma_T}{c}\right)^2 \int \frac{ds}{s^2 a^4} \frac{P_{q_\perp}(k = l/s, z)}{2}$$

(See Park et al. 2013 for the derivation)

kSZ power spectrum is given by projecting the transverse component of ionized momentum field along the line-of-sight.

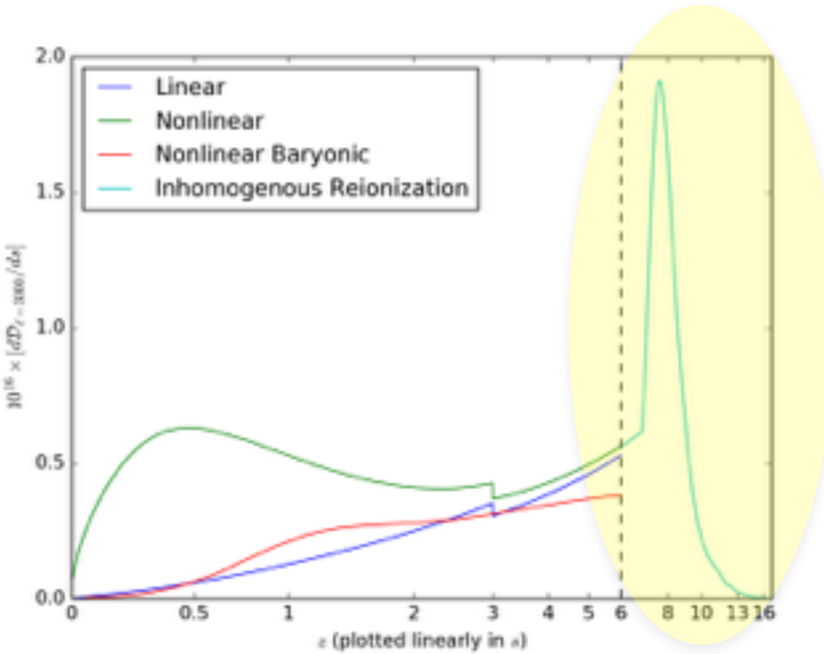
Reionization vs Post-reionization

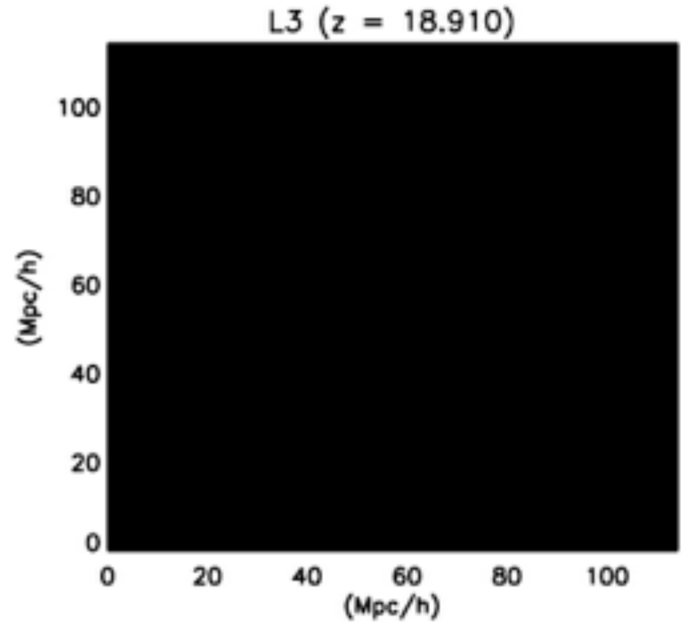


$\mathbf{q} \equiv X(1+\delta)\mathbf{v}$

1) **At $z > 6$, inhomogeneously ionized IGM gives a huge boost to the signal.**
2) **At $z < 6$, non-linear growth of structure and baryonic physics make huge impacts on the signal.**

Part 1) Reionization Signal





$$\mathbf{q} = X(1+\delta)\mathbf{v} = X(1+\delta_{lin})\mathbf{v}_{lin}$$

1) **Inhomogeneous ionization strongly enhances the signal.**
2) **The signal is highly dependent on the properties of the source.**

(Park et al. 2013)

Part 1) Properties of Ionizing Sources During EoR

- High-mass Atomic Cooling Halos: $M > 10^9\ M_\odot$

  Never suppressed.

- Low-mass Atomic Cooling Halos: $10^8\ M_\odot < M < 10^9\ M_\odot$

  Suppressed when surrounding region is ionized.

- Minihalos: $M < 10^8\ M_\odot$

  Suppressed when surrounding region is ionized or exposed to H_2 dissociating radiation.

1-1) Models with and without Low-mass Galaxies

- **High-mass Atomic Cooling Halos: $M > 10^9\ M_\odot$**

  Never suppressed.

- **Low-mass Atomic Cooling Halos: $10^8\ M_\odot < M < 10^9\ M_\odot$**

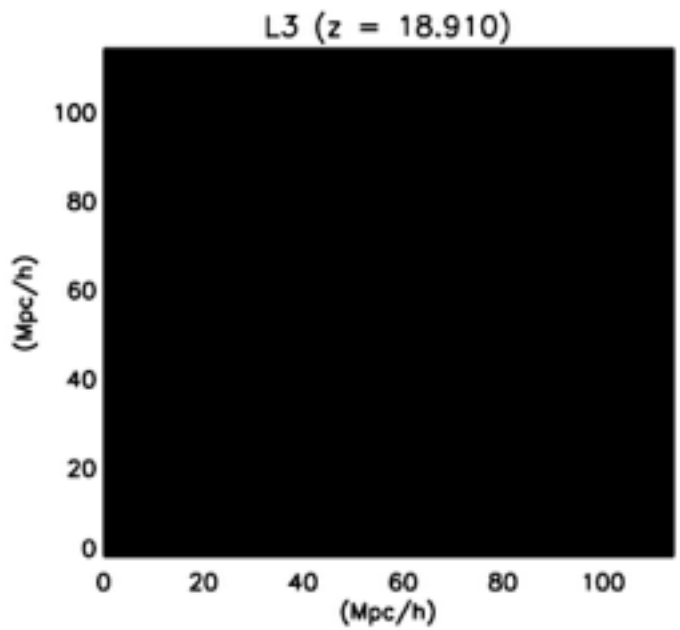
  Suppressed when surrounding region is ionized.

- Minihalos: $M < 10^8\ M_\odot$

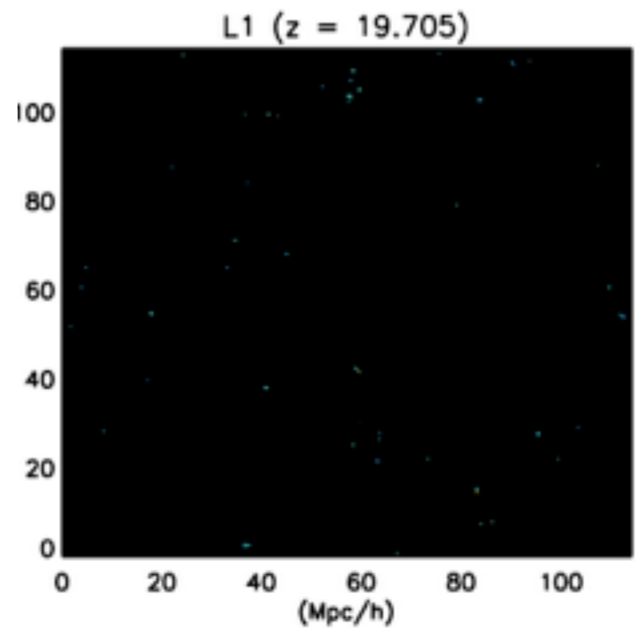
  Suppressed when surrounding region is ionized or exposed to H_2 dissociating radiation.

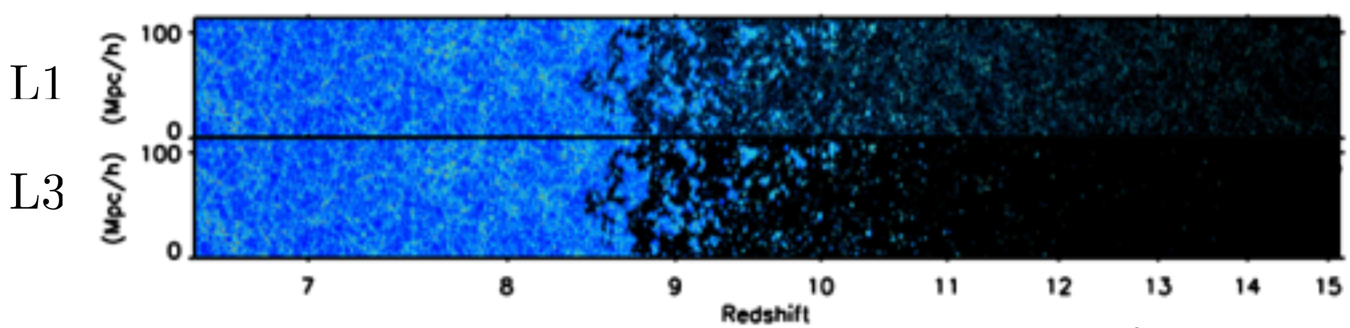
1-1) Models with and without Low-mass Galaxies

High-mass sources only



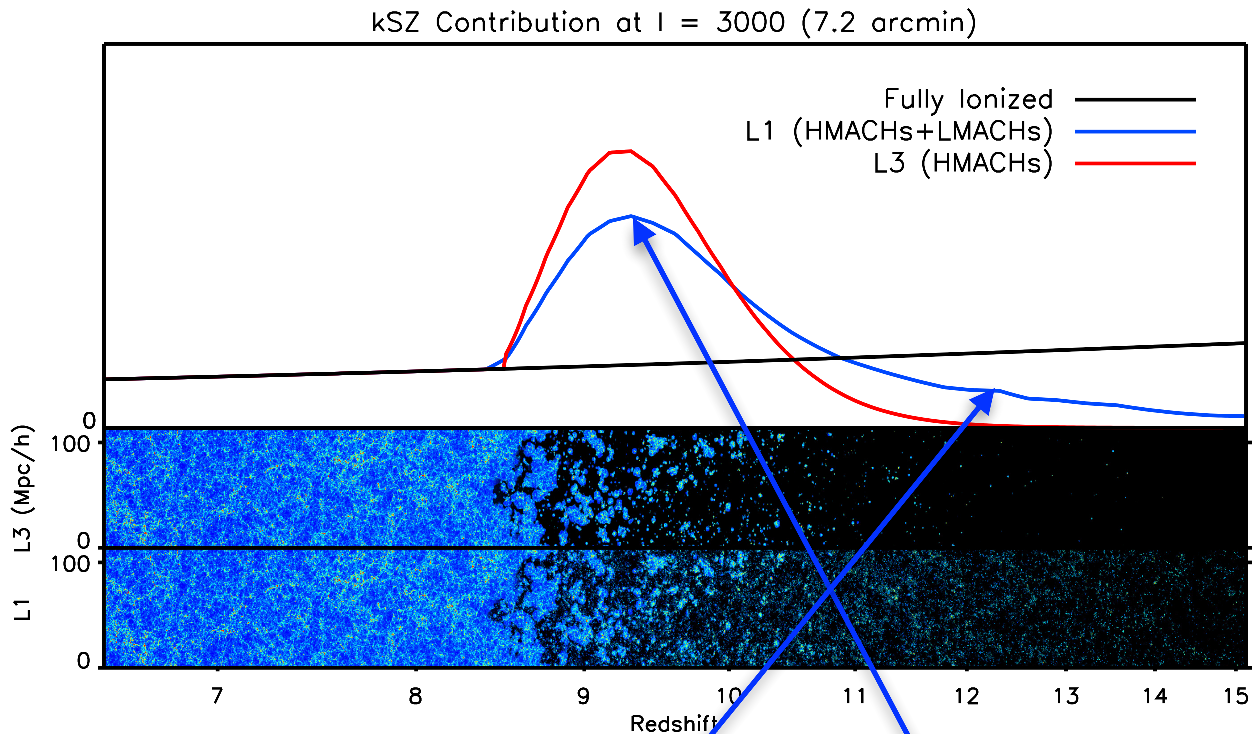
Low-mass sources added





(Park et al. 2013)

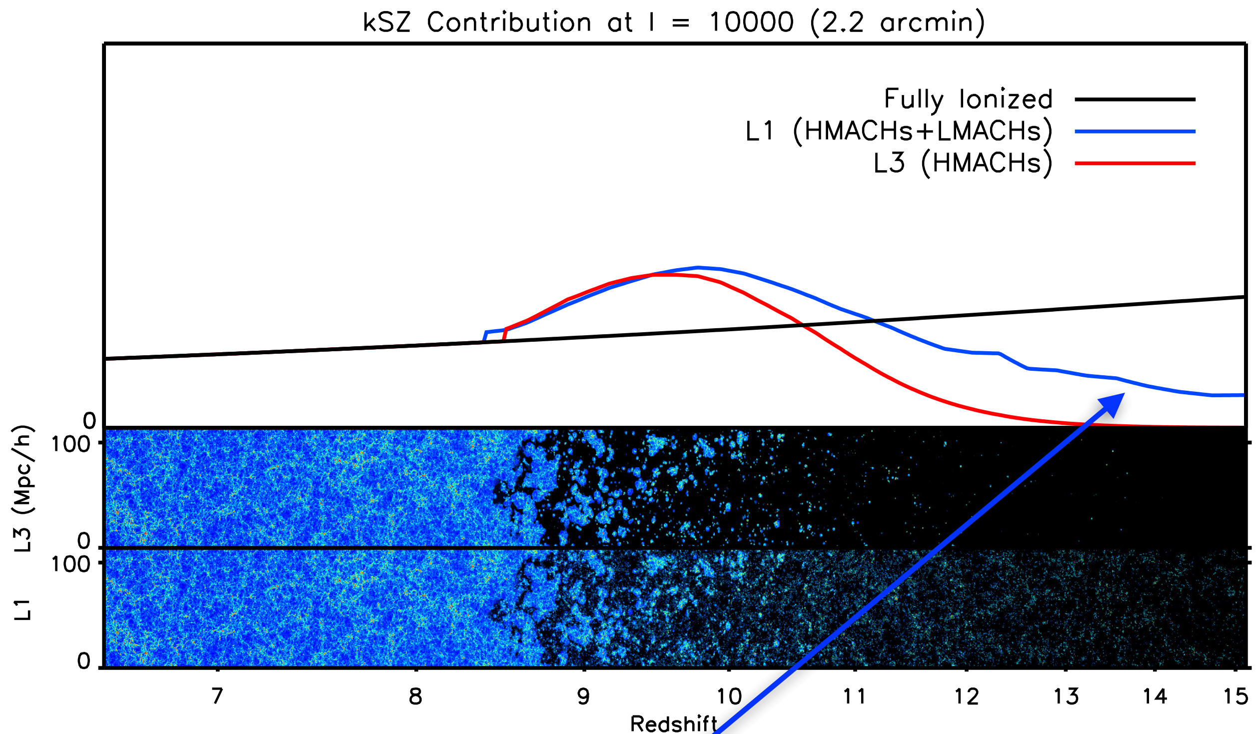
1-1) Models with and without Low-mass Galaxies (l = 3000)



Including low-mass galaxies give a <u>lower signal at late time</u> and <u>higher signal at early time</u>. Both effects nearly cancel each other out to give a similar total signal, 0.9 μK².

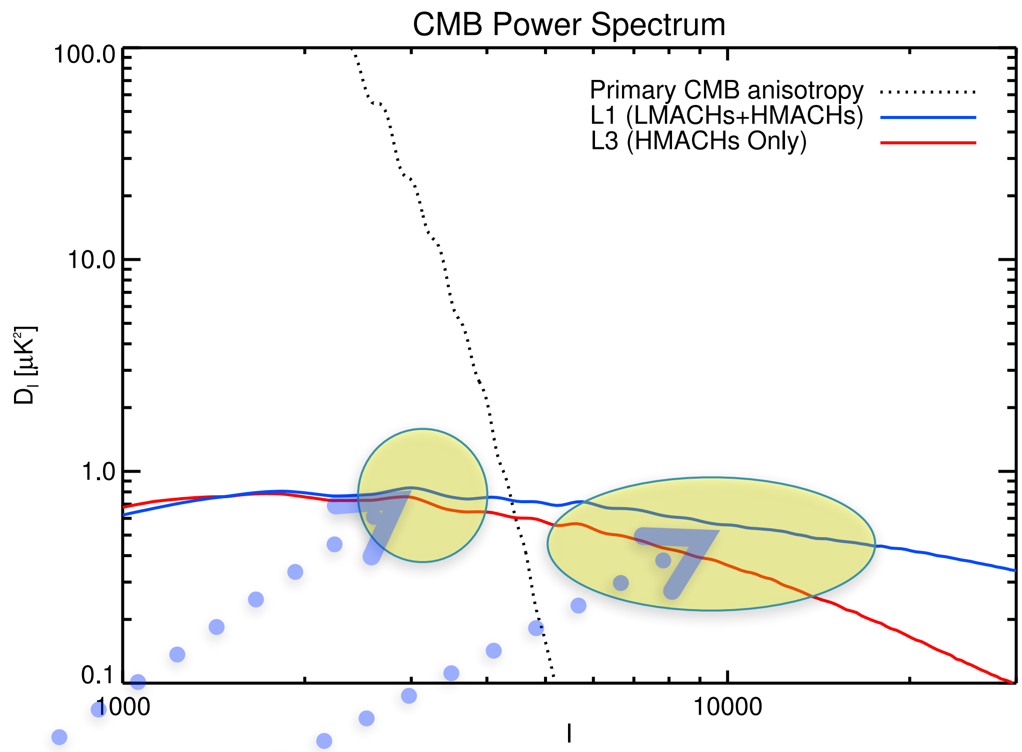
(Park et al. 2013)

1-1) Models with and without Low-mass Galaxies (*l* = 10000)



The low-mass galaxies have a larger impact in smaller angular scales.

Result 1-1) Impact of low-mass galaxies on the kSZ Power Spectrum



- **At *l* = 3000 (~3 arcmin), adding low-mass galaxies make kSZ signal only 5% larger.**
- **At larger *l*'s, the difference is larger.**

(Park et al. 2013)

1-2) Models with and without Mihihalo Galaxies

- **High-mass Atomic Cooling Halos: $M > 10^9\ M_\odot$**

  Never suppressed.

- Low-mass Atomic Cooling Halos: $10^8\ M_\odot < M < 10^9\ M_\odot$

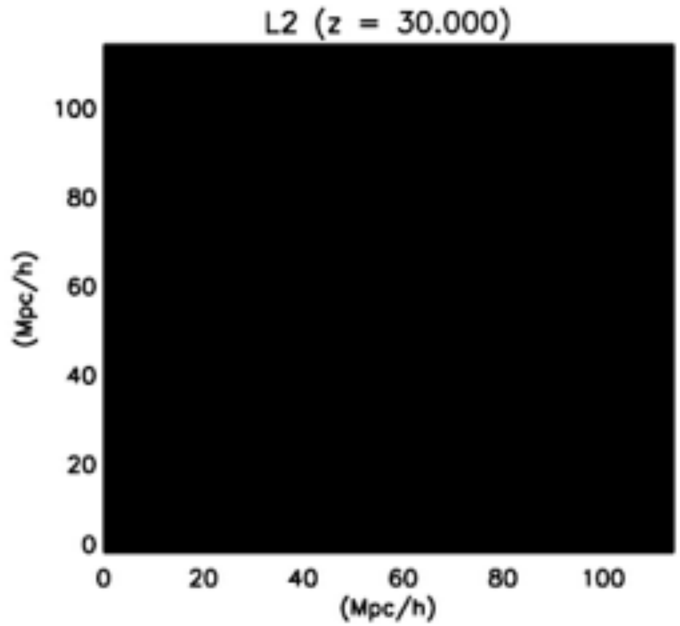
  Suppressed when surrounding region is ionized.

- **Minihalos: $M < 10^8\ M_\odot$**

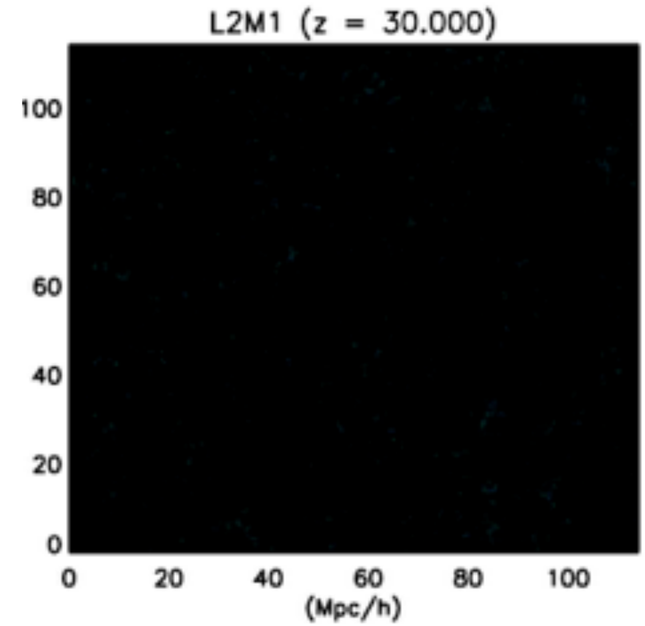
  Suppressed when surrounding region is ionized or exposed to H_2 dissociating radiation.

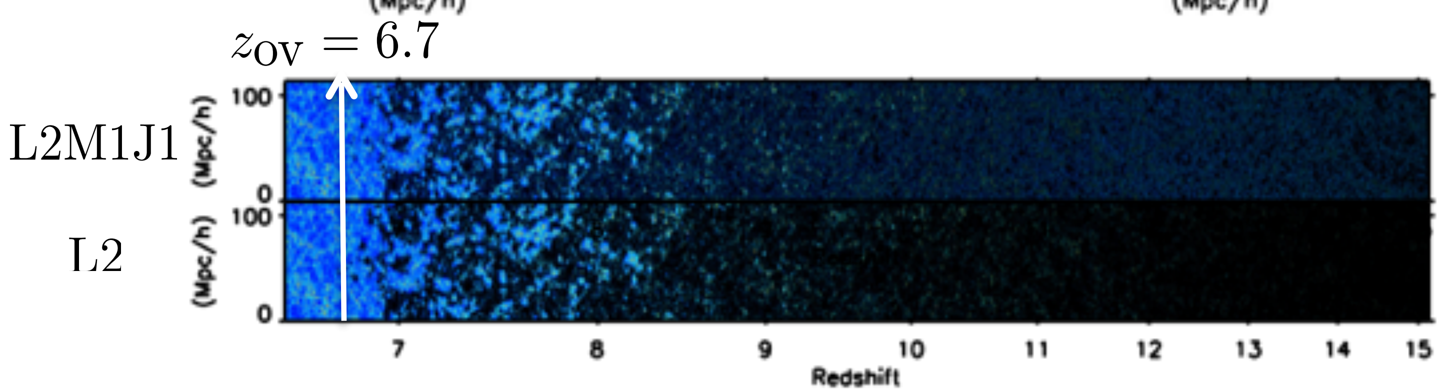
1-2) Models with and without Mihihalo Galaxies

HMACHs and LMACHs

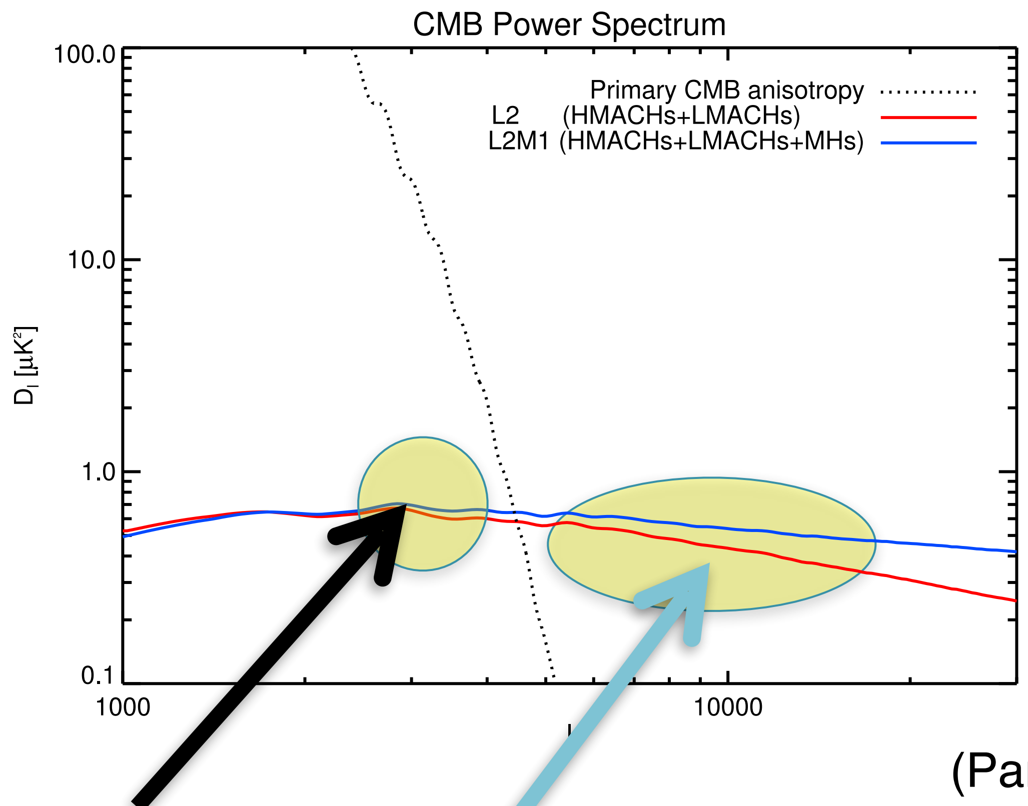


HMACHs, LMACHs and MHs





Result 1-2) Impact of Minihalo galaxies on kSZ Power Spectrum

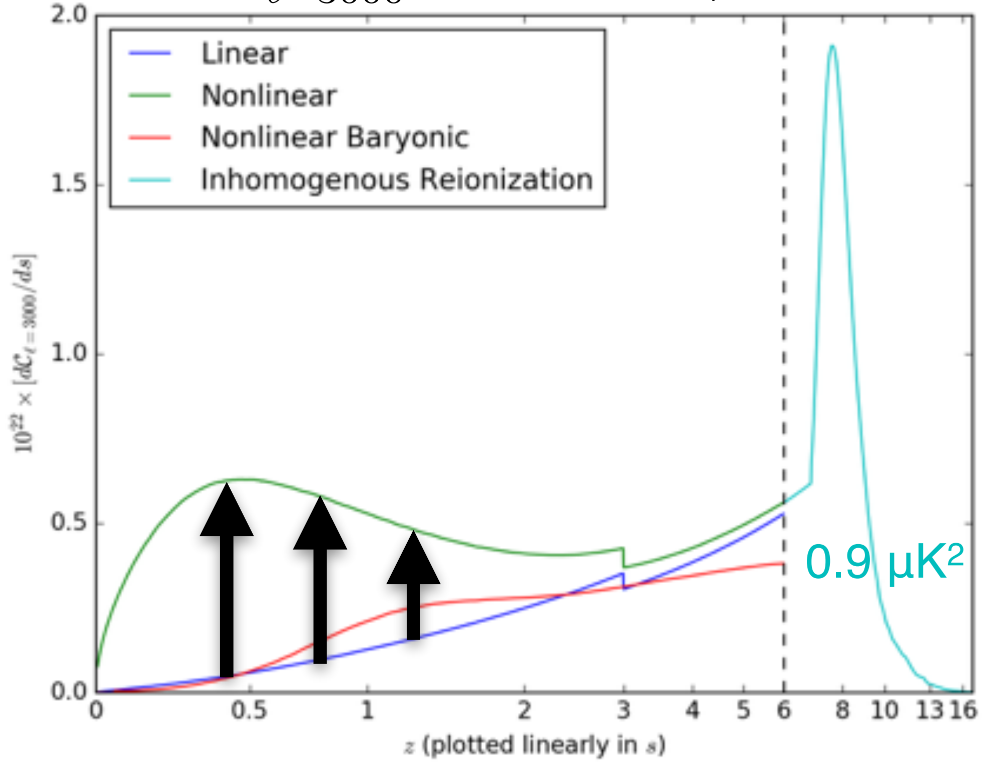


(Park et al. 2013)

- **At l = 3000, adding MHs make kSZ signal only 3% larger.**
- **At larger l's, the difference is larger.**

Part 2) Impact of Nonlinear Growth of Structure on the Post-reionization Signal

$$\mathcal{D}^{\rm SPT}_{\ell=3000} = 2.9 \pm 1.3\ \mu K^2$$



$$\mathbf{q} = X(1+\delta)\mathbf{v} = (1+\delta_{nl})\mathbf{v}_{nl}$$

Ionized fraction (X) is one everywhere. But, density (δ) and velocity (v) are in the nonlinear regime.

Expressing $P_{q,perp}$ in terms of density(δ) and velocity($\mathbf{v}$)

In the post-reionization epoch, IGM is fully ionized to a good approximation. Therefore, we can set X = 1.

$$\mathbf{q} = \mathbf{v}(1+\delta) = \mathbf{v} + \mathbf{v}\delta$$

$$\tilde{\mathbf{q}}(\mathbf{k}) = \tilde{\mathbf{v}}(\mathbf{k}) + \int \frac{d^3k'}{(2\pi)^3}\tilde{\mathbf{v}}(\mathbf{k}')\tilde{\delta}(\mathbf{k}-\mathbf{k}')$$

$$\tilde{\mathbf{q}}_\perp(\mathbf{k}) = \cancel{\tilde{\mathbf{v}}_\perp(\mathbf{k})} + \boxed{\left(\int \frac{d^3k'}{(2\pi)^3}\tilde{\mathbf{v}}(\mathbf{k}')\tilde{\delta}(\mathbf{k}-\mathbf{k}')\right)_\perp}$$

Since the transverse mode is negligibly small for velocity, the transverse momentum field has only a 2nd order term.

$P_{q,perp}$ in terms of δ and **v**

$$(2\pi)^3 P_{q_\perp}(\mathbf{k_1})\delta_D(\mathbf{k_1}+\mathbf{k_2}) = \langle \tilde{\mathbf{q}}_\perp(\mathbf{k_1})\tilde{\mathbf{q}}_\perp(\mathbf{k_2})\rangle$$

$$= \left\langle \int \frac{d^3k'}{(2\pi)^3}\tilde{\mathbf{v}}_\perp(\mathbf{k}')\delta(\mathbf{k_1}-\mathbf{k}')\cdot\int \frac{d^3k'}{(2\pi)^3}\tilde{\mathbf{v}}_\perp(\mathbf{k}')\delta(\mathbf{k_2}-\mathbf{k}')\right\rangle$$

$$P_{q_\perp} = \sum_{i=1}^{3}\left[1-(\hat{k}^i)^2\right]P_q^{ii}$$

where

$$(2\pi)^3 P_q^{ij}(k)\delta_D(\mathbf{k_1}+\mathbf{k_2})$$

$$= \int \frac{d^3k'}{(2\pi)^3}\int \frac{d^3k''}{(2\pi)^3}\left\langle \tilde{\delta}(\mathbf{k_1}-\mathbf{k}')\tilde{\delta}(\mathbf{k_2}-\mathbf{k}'')\tilde{v}^i(\mathbf{k}')\tilde{v}^j(\mathbf{k}'')\right\rangle$$

$P_{q,\mathrm{perp}}$ in terms of δ and **v**

Using Wick's theorem
(<ABCD> = <AB><CD> + <AC><BD> + <AD><BC>),
the 4th order bracket can be decomposed into pair brackets.

$$
\begin{aligned}
&\left\langle \tilde{\delta}(\mathbf{k_1}-\mathbf{k}')\tilde{v}^i(\mathbf{k}')\tilde{\delta}(\mathbf{k_2}-\mathbf{k}'')\tilde{v}^j(\mathbf{k}'')\right\rangle \\
&= \left\langle \tilde{\delta}(\mathbf{k_1}-\mathbf{k}')\tilde{v}^i(\mathbf{k}')\right\rangle \left\langle \tilde{\delta}(\mathbf{k_2}-\mathbf{k}'')\tilde{v}^j(\mathbf{k}'')\right\rangle \\
&+ \left\langle \tilde{\delta}(\mathbf{k_1}-\mathbf{k}')\tilde{\delta}(\mathbf{k_2}-\mathbf{k}'')\right\rangle \left\langle \tilde{v}^i(\mathbf{k}')\tilde{v}^j(\mathbf{k}'')\right\rangle \\
&+ \left\langle \tilde{\delta}(\mathbf{k_1}-\mathbf{k}')\tilde{v}^j(\mathbf{k}'')\right\rangle \left\langle \tilde{\delta}(\mathbf{k_2}-\mathbf{k}'')\tilde{v}^i(\mathbf{k}')\right\rangle
\end{aligned}
$$

$P_{q,\mathrm{perp}}$ in terms of δ and **v**

$$\left\langle \tilde{\delta}(\mathbf{k_1}-\mathbf{k}')\tilde{v}^i(\mathbf{k}')\tilde{\delta}(\mathbf{k_2}-\mathbf{k}'')\tilde{v}^j(\mathbf{k}'')\right\rangle$$

$$= \left\langle \tilde{\delta}(\mathbf{k_1}-\mathbf{k}')\tilde{v}^i(\mathbf{k}')\right\rangle\left\langle\tilde{\delta}(\mathbf{k_2}-\mathbf{k}'')\tilde{v}^j(\mathbf{k}'')\right\rangle$$

$$+ \left\langle \tilde{\delta}(\mathbf{k_1}-\mathbf{k}')\tilde{\delta}(\mathbf{k_2}-\mathbf{k}'')\right\rangle\left\langle\tilde{v}^i(\mathbf{k}')\tilde{v}^j(\mathbf{k}'')\right\rangle$$

$$+ \left\langle \tilde{\delta}(\mathbf{k_1}-\mathbf{k}')\tilde{v}^j(\mathbf{k}'')\right\rangle\left\langle\tilde{\delta}(\mathbf{k_2}-\mathbf{k}'')\tilde{v}^i(\mathbf{k}')\right\rangle$$

$$P_{q_\perp} = \int \frac{d^3k'}{(2\pi)^3}\left(1-\mu'^2\right)$$

$$\left[P_{\delta\delta}(|\mathbf{k}-\mathbf{k}'|)P_{vv}(\mathbf{k}') - \frac{k'}{|\mathbf{k}-\mathbf{k}'|}P_{\delta v}(|\mathbf{k}-\mathbf{k}'|)P_{\delta v}(\mathbf{k}')\right]$$

$(\mu' \equiv \hat{k}\cdot\hat{k}')$

(Ma & Fry 2002)

$P_{q,perp}$ in terms of δ and **v**

$$P_{q_\perp} = \int \frac{d^3k'}{(2\pi)^3} \left(1 - {\mu'}^2\right)$$

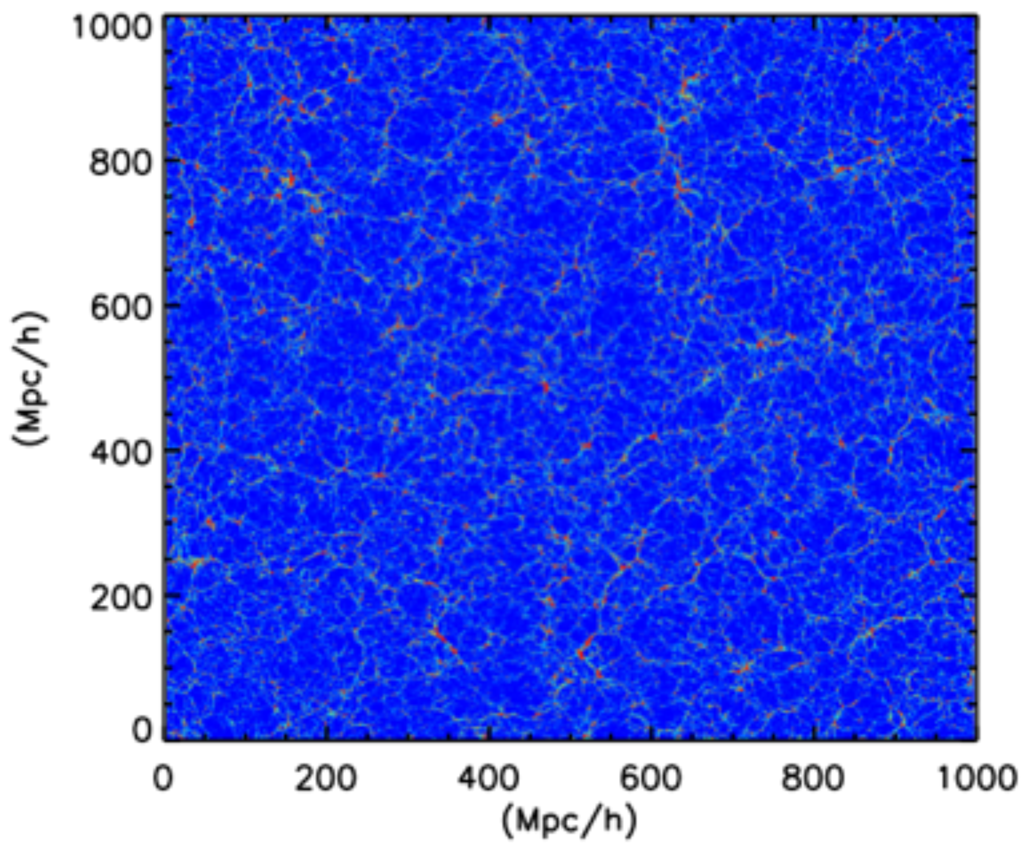
$$\left[P_{\delta\delta}(|\mathbf{k} - \mathbf{k}'|) P_{vv}(\mathbf{k}') - \frac{k'}{|\mathbf{k} - \mathbf{k}'|} P_{\delta v}(|\mathbf{k} - \mathbf{k}'|) P_{\delta v}(\mathbf{k}')\right]$$

$$(\mu' \equiv \hat{k} \cdot \hat{k}')$$

(Ma & Fry 2002)

Does this expression really hold in the nonlinear regime?

Testing the analytic expression of $P_{q,perp}$



Code : CubeP3M

Size : 1 Gpc/h

# of ptls : 3456^3

We use *N*-body simulation data to test the non-linear expression

$P_{q,perp}$ in terms of δ and **v**

$$P_{q_\perp} = \int \frac{d^3k'}{(2\pi)^3}\left(1-\mu'^2\right)$$

$$\left[P_{\delta\delta}(|\mathbf{k}-\mathbf{k}'|)P_{vv}(\mathbf{k}') - \frac{k'}{|\mathbf{k}-\mathbf{k}'|}P_{\delta v}(|\mathbf{k}-\mathbf{k}'|)P_{\delta v}(\mathbf{k}')\right]$$

$(\mu' \equiv \hat{k}\cdot\hat{k}')$

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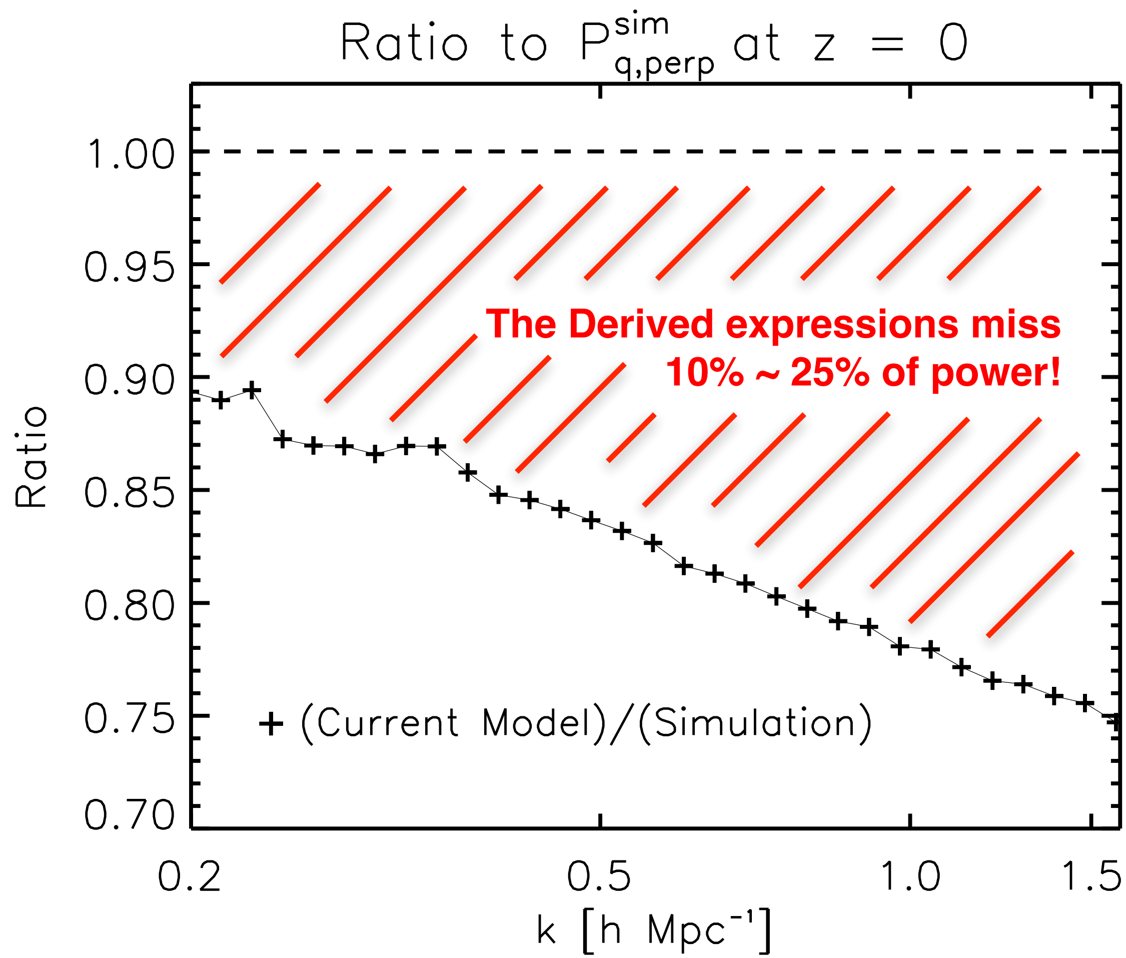
Testing the analytic expression of $P_{q,perp}$



(Park et al. 2016)

What is wrong with the analytic expression?

Wick's theorem works only for gaussian fields. In the nonlinear regime where density and velocity becomes non-gaussian, we need a correction term called "Connected Moment".

$$\left\langle \tilde{\delta}(\mathbf{k_1}-\mathbf{k}')\tilde{v}^i(\mathbf{k}')\tilde{\delta}(\mathbf{k_2}-\mathbf{k}'')\tilde{v}^j(\mathbf{k}'') \right\rangle$$

$$= \left\langle \tilde{\delta}(\mathbf{k_1}-\mathbf{k}')\tilde{v}^i(\mathbf{k}') \right\rangle \left\langle \tilde{\delta}(\mathbf{k_2}-\mathbf{k}'')\tilde{v}^j(\mathbf{k}'') \right\rangle$$

$$+ \left\langle \tilde{\delta}(\mathbf{k_1}-\mathbf{k}')\tilde{\delta}(\mathbf{k_2}-\mathbf{k}'') \right\rangle \langle \tilde{v}^i(\mathbf{k}')\tilde{v}^j(\mathbf{k}'') \rangle$$

$$+ \left\langle \tilde{\delta}(\mathbf{k_1}-\mathbf{k}')\tilde{v}^j(\mathbf{k}'') \right\rangle \left\langle \tilde{\delta}(\mathbf{k_2}-\mathbf{k}'')\tilde{v}^i(\mathbf{k}') \right\rangle$$

$$+(\text{Connected Moment})$$

Perturbation Theory Expansion

$$\delta = \delta^{(1)} + \delta^{(2)} + ...$$

$$\delta^{(2)}(\mathbf{k}) = \int \frac{d^3k_a}{(2\pi)^3} \int d^3k_b \; \delta^{(1)}(\mathbf{k_a})\delta^{(1)}(\mathbf{k_b})F_2(\mathbf{k_a},\mathbf{k_b})\delta_D(\mathbf{k}-\mathbf{k_a}-\mathbf{k_b})$$

$$F_2(\mathbf{q_1},\mathbf{q_2}) = \frac{5}{7} + \frac{1}{2}\frac{\mathbf{q_1}\cdot\mathbf{q_2}}{q_1 q_2} + \frac{2}{7}\left(\frac{\mathbf{q_1}\cdot\mathbf{q_2}}{q_1 q_2}\right)^2$$

$$\theta \equiv \nabla\cdot\mathbf{v} = \theta^{(1)} + \theta^{(2)} + ...$$

$$\theta^{(2)}(\mathbf{k}) = -\int \frac{d^3k_a}{(2\pi)^3} \int d^3k_b \; \delta^{(1)}(\mathbf{k_a})\delta^{(1)}(\mathbf{k_b})G_2(\mathbf{k_a},\mathbf{k_b})\delta_D(\mathbf{k}-\mathbf{k_a}-\mathbf{k_b})$$

$$G_2(\mathbf{q_1},\mathbf{q_2}) = \frac{3}{7} + \frac{1}{2}\frac{\mathbf{q_1}\cdot\mathbf{q_2}}{q_1 q_2} + \frac{4}{7}\left(\frac{\mathbf{q_1}\cdot\mathbf{q_2}}{q_1 q_2}\right)^2$$

Nonlinear terms can be expressed in terms of linear terms using perturbation theory

P_q: Next-to-leading order

$$\left\langle \tilde{\delta}(\mathbf{k_1})\tilde{\delta}(\mathbf{k_2})\tilde{\mathbf{v}}(\mathbf{k_3})\tilde{\mathbf{v}}(\mathbf{k_4})\right\rangle = -\dot{a}^2\left(\frac{\hat{\mathbf{k}}_3}{k_3}\right)\left(\frac{\hat{\mathbf{k}}_4}{k_4}\right)\left\langle \tilde{\delta}(\mathbf{k_1})\tilde{\delta}(\mathbf{k_2})\tilde{\theta}(\mathbf{k_3})\tilde{\theta}(\mathbf{k_4})\right\rangle$$

$$\delta = \delta^{(1)} + \delta^{(2)} + \dots \qquad \theta \equiv i\nabla\cdot\mathbf{v} = \theta^{(1)} + \theta^{(2)} + \dots$$

Next-to-leading order of $\left\langle \tilde{\delta}(\mathbf{k_1})\tilde{\delta}(\mathbf{k_2})\tilde{\theta}(\mathbf{k_3})\tilde{\theta}(\mathbf{k_4})\right\rangle$ is

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$$\left\langle \tilde{\delta}^{(\mathbf{2})}(\mathbf{k_1})\tilde{\delta}^{(\mathbf{2})}(\mathbf{k_2})\tilde{\theta}^{(1)}(\mathbf{k_3})\tilde{\theta}^{(1)}(\mathbf{k_4})\right\rangle$$

$$\left\langle \tilde{\delta}^{(\mathbf{2})}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_2})\tilde{\theta}^{(\mathbf{2})}(\mathbf{k_3})\tilde{\theta}^{(1)}(\mathbf{k_4})\right\rangle$$

$$\left\langle \tilde{\delta}^{(\mathbf{2})}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_2})\tilde{\theta}^{(1)}(\mathbf{k_3})\tilde{\theta}^{(\mathbf{2})}(\mathbf{k_4})\right\rangle$$

$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(\mathbf{2})}(\mathbf{k_2})\tilde{\theta}^{(\mathbf{2})}(\mathbf{k_3})\tilde{\theta}^{(1)}(\mathbf{k_4})\right\rangle$$

$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(\mathbf{2})}(\mathbf{k_2})\tilde{\theta}^{(1)}(\mathbf{k_3})\tilde{\theta}^{(\mathbf{2})}(\mathbf{k_4})\right\rangle$$

$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_2})\tilde{\theta}^{(\mathbf{2})}(\mathbf{k_3})\tilde{\theta}^{(\mathbf{2})}(\mathbf{k_4})\right\rangle$$

$+$

$$\left\langle \tilde{\delta}^{(\mathbf{3})}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_2})\tilde{\theta}^{(1)}(\mathbf{k_3})\tilde{\theta}^{(1)}(\mathbf{k_4})\right\rangle$$

$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(\mathbf{3})}(\mathbf{k_2})\tilde{\theta}^{(1)}(\mathbf{k_3})\tilde{\theta}^{(1)}(\mathbf{k_4})\right\rangle$$

$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_2})\tilde{\theta}^{(\mathbf{3})}(\mathbf{k_3})\tilde{\theta}^{(1)}(\mathbf{k_4})\right\rangle$$

$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_2})\tilde{\theta}^{(1)}(\mathbf{k_3})\tilde{\theta}^{(\mathbf{3})}(\mathbf{k_4})\right\rangle$$

Connected Moment of $P_{q,perp}$
Next-to-leading order

$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(2)}(\mathbf{k_2})\tilde{\theta}^{(2)}(\mathbf{k_3})\tilde{\theta}^{(1)}(\mathbf{k_4})\right\rangle$$

$$= \int \frac{d^3\mathbf{k_{2,a}}}{(2\pi)^3}\int d^3\mathbf{k_{2,b}}\int \frac{d^3\mathbf{k_{3,a}}}{(2\pi)^3}\int d^3\mathbf{k_{3,b}}$$

$$F_2^{(s)}(\mathbf{k_{2,a}},\mathbf{k_{2,b}})G_2^{(s)}(\mathbf{k_{3,a}},\mathbf{k_{3,b}})\delta_{\mathrm{D}}(\mathbf{k_2}-\mathbf{k_{2,a}}-\mathbf{k_{2,b}})\delta_{\mathrm{D}}(\mathbf{k_3}-\mathbf{k_{3,a}}-\mathbf{k_{3,b}})$$

$$\underline{\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle}$$

$$= \int \frac{d^3\mathbf{k_{2,a}}}{(2\pi)^3}\int d^3\mathbf{k_{2,b}}\int \frac{d^3\mathbf{k_{3,a}}}{(2\pi)^3}\int d^3\mathbf{k_{3,b}}$$

$$F_2^{(s)}(\mathbf{k_{2,a}},\mathbf{k_{2,b}})G_2^{(s)}(\mathbf{k_{3,a}},\mathbf{k_{3,b}})\delta_{\mathrm{D}}(\mathbf{k_2}-\mathbf{k_{2,a}}-\mathbf{k_{2,b}})\delta_{\mathrm{D}}(\mathbf{k_3}-\mathbf{k_{3,a}}-\mathbf{k_{3,b}})$$

$$\underline{\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\right\rangle\left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\right\rangle\left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle}$$

.

. 15 terms in total

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$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(2)}(\mathbf{k_2})\tilde{\theta}^{(2)}(\mathbf{k_3})\tilde{\theta}^{(1)}(\mathbf{k_4}) \right\rangle$$

$$= \int \frac{d^3\mathbf{k_{2,a}}}{(2\pi)^3} \int d^3\mathbf{k_{2,b}} \int \frac{d^3\mathbf{k_{3,a}}}{(2\pi)^3} \int d^3\mathbf{k_{3,b}}$$

$$F_2^{(s)}(\mathbf{k_{2,a}}, \mathbf{k_{2,b}}) G_2^{(s)}(\mathbf{k_{3,a}}, \mathbf{k_{3,b}}) \delta_{\mathrm{D}}(\mathbf{k_2} - \mathbf{k_{2,a}} - \mathbf{k_{2,b}}) \delta_{\mathrm{D}}(\mathbf{k_3} - \mathbf{k_{3,a}} - \mathbf{k_{3,b}})$$

$$\begin{gathered}
\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{2,a}}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\tilde{\delta}^{(1)}(\mathbf{k_{3,a}}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\tilde{\delta}^{(1)}(\mathbf{k_4}) \right\rangle \\
\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{2,a}}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_4}) \right\rangle \\
\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{2,a}}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\tilde{\delta}^{(1)}(\mathbf{k_4}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}}) \right\rangle \\
\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{3,a}}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\tilde{\delta}^{(1)}(\mathbf{k_4}) \right\rangle \\
\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_4}) \right\rangle \\
\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_4}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}}) \right\rangle \\
\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{3,a}}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\tilde{\delta}^{(1)}(\mathbf{k_4}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}}) \right\rangle \\
\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{3,a}}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_4}) \right\rangle \\
\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{3,a}}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\tilde{\delta}^{(1)}(\mathbf{k_4}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}}) \right\rangle \\
\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_4}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}}) \right\rangle \\
\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_4}) \right\rangle \\
\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_4}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}}) \right\rangle \\
\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_4}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}}) \right\rangle \\
\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_4}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{3,a}}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}}) \right\rangle \\
\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_4}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}}) \right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}}) \right\rangle
\end{gathered}$$

$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(2)}(\mathbf{k_2})\tilde{\theta}^{(2)}(\mathbf{k_3})\tilde{\theta}^{(1)}(\mathbf{k_4})\right\rangle$$

$$\begin{aligned}
&\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(2)}(\mathbf{k_2})\tilde{\theta}^{(2)}(\mathbf{k_3})\tilde{\theta}^{(1)}(\mathbf{k_4})\right\rangle_{uc} \\
&= \left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(2)}(\mathbf{k_2})\right\rangle \left\langle \tilde{\theta}^{(2)}(\mathbf{k_3})\tilde{\theta}^{(1)}(\mathbf{k_4})\right\rangle \\
&+ \left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(2)}(\mathbf{k_3})\right\rangle \left\langle \tilde{\theta}^{(2)}(\mathbf{k_2})\tilde{\theta}^{(1)}(\mathbf{k_4})\right\rangle \\
&+ \left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \left\langle \tilde{\theta}^{(2)}(\mathbf{k_2})\tilde{\theta}^{(2)}(\mathbf{k_3})\right\rangle
\end{aligned}$$

$$= \int \frac{d^3\mathbf{k_{2,a}}}{(2\pi)^3}\int d^3\mathbf{k_{2,b}}\int \frac{d^3\mathbf{k_{3,a}}}{(2\pi)^3}\int d^3\mathbf{k_{3,b}}$$

$$F_2^{(s)}(\mathbf{k_{2,a}},\mathbf{k_{2,b}})G_2^{(s)}(\mathbf{k_{3,a}},\mathbf{k_{3,b}})\delta_{\mathrm{D}}(\mathbf{k_2}-\mathbf{k_{2,a}}-\mathbf{k_{2,b}})\delta_{\mathrm{D}}(\mathbf{k_3}-\mathbf{k_{3,a}}-\mathbf{k_{3,b}})$$

$$\begin{aligned}
&\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \\
&\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \\
&\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\right\rangle \\
&\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \\
&\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \\
&\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\right\rangle \\
&\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\right\rangle \\
&\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \\
&\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\right\rangle \\
&\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\right\rangle \\
&\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \\
&\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\right\rangle \\
&\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\right\rangle \\
&\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\right\rangle \\
&\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\right\rangle
\end{aligned}$$

$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(2)}(\mathbf{k_2})\tilde{\theta}^{(2)}(\mathbf{k_3})\tilde{\theta}^{(1)}(\mathbf{k_4})\right\rangle$$

$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(2)}(\mathbf{k_2})\tilde{\theta}^{(2)}(\mathbf{k_3})\tilde{\theta}^{(1)}(\mathbf{k_4})\right\rangle_{uc} = \left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \left\langle \tilde{\theta}^{(2)}(\mathbf{k_2})\tilde{\theta}^{(2)}(\mathbf{k_3})\right\rangle$$

$$= \int \frac{d^3\mathbf{k_{2,a}}}{(2\pi)^3} \int d^3\mathbf{k_{2,b}} \int \frac{d^3\mathbf{k_{3,a}}}{(2\pi)^3} \int d^3\mathbf{k_{3,b}}$$
$$F_2^{(s)}(\mathbf{k_{2,a}}, \mathbf{k_{2,b}})G_2^{(s)}(\mathbf{k_{3,a}}, \mathbf{k_{3,b}})\delta_{\mathrm{D}}(\mathbf{k_2} - \mathbf{k_{2,a}} - \mathbf{k_{2,b}})\delta_{\mathrm{D}}(\mathbf{k_3} - \mathbf{k_{3,a}} - \mathbf{k_{3,b}})$$

$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle$$
$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle$$
$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\right\rangle$$
$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle$$
$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle$$
$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\right\rangle$$
$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\right\rangle$$
$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle$$
$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\right\rangle$$
$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\right\rangle$$
$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle$$
$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\right\rangle$$

Connected Moment

$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\right\rangle$$
$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\right\rangle$$
$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\right\rangle$$

Unconnected Moment

$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\right\rangle$$
$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\right\rangle$$
$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_4})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{2,a}})\tilde{\delta}^{(1)}(\mathbf{k_{3,b}})\right\rangle \left\langle \tilde{\delta}^{(1)}(\mathbf{k_{3,a}})\tilde{\delta}^{(1)}(\mathbf{k_{2,b}})\right\rangle$$

P_q: Next-to-leading order

$$\left\langle \tilde{\delta}(\mathbf{k_1})\tilde{\delta}(\mathbf{k_2})\tilde{\mathbf{v}}(\mathbf{k_3})\tilde{\mathbf{v}}(\mathbf{k_4})\right\rangle = -\dot{a}^2\left(\frac{\hat{\mathbf{k}}_3}{k_3}\right)\left(\frac{\hat{\mathbf{k}}_4}{k_4}\right)\left\langle \tilde{\delta}(\mathbf{k_1})\tilde{\delta}(\mathbf{k_2})\tilde{\theta}(\mathbf{k_3})\tilde{\theta}(\mathbf{k_4})\right\rangle$$

$$\delta = \delta^{(1)} + \delta^{(2)} + \ldots \qquad \theta \equiv i\nabla\cdot\mathbf{v} = \theta^{(1)} + \theta^{(2)} + \ldots$$

Next-to-leading order of $\left\langle \tilde{\delta}(\mathbf{k_1})\tilde{\delta}(\mathbf{k_2})\tilde{\theta}(\mathbf{k_3})\tilde{\theta}(\mathbf{k_4})\right\rangle$ is

---

$$\left\langle \tilde{\delta}^{(\mathbf{2})}(\mathbf{k_1})\tilde{\delta}^{(\mathbf{2})}(\mathbf{k_2})\tilde{\theta}^{(1)}(\mathbf{k_3})\tilde{\theta}^{(1)}(\mathbf{k_4})\right\rangle$$

$$\left\langle \tilde{\delta}^{(\mathbf{2})}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_2})\tilde{\theta}^{(\mathbf{2})}(\mathbf{k_3})\tilde{\theta}^{(1)}(\mathbf{k_4})\right\rangle$$

$$\left\langle \tilde{\delta}^{(\mathbf{2})}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_2})\tilde{\theta}^{(1)}(\mathbf{k_3})\tilde{\theta}^{(\mathbf{2})}(\mathbf{k_4})\right\rangle$$

$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(\mathbf{2})}(\mathbf{k_2})\tilde{\theta}^{(\mathbf{2})}(\mathbf{k_3})\tilde{\theta}^{(1)}(\mathbf{k_4})\right\rangle$$

$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(\mathbf{2})}(\mathbf{k_2})\tilde{\theta}^{(1)}(\mathbf{k_3})\tilde{\theta}^{(\mathbf{2})}(\mathbf{k_4})\right\rangle$$

$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_2})\tilde{\theta}^{(\mathbf{2})}(\mathbf{k_3})\tilde{\theta}^{(\mathbf{2})}(\mathbf{k_4})\right\rangle$$

+

$$\left\langle \tilde{\delta}^{(\mathbf{3})}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_2})\tilde{\theta}^{(1)}(\mathbf{k_3})\tilde{\theta}^{(1)}(\mathbf{k_4})\right\rangle$$

$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(\mathbf{3})}(\mathbf{k_2})\tilde{\theta}^{(1)}(\mathbf{k_3})\tilde{\theta}^{(1)}(\mathbf{k_4})\right\rangle$$

$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_2})\tilde{\theta}^{(\mathbf{3})}(\mathbf{k_3})\tilde{\theta}^{(1)}(\mathbf{k_4})\right\rangle$$

$$\left\langle \tilde{\delta}^{(1)}(\mathbf{k_1})\tilde{\delta}^{(1)}(\mathbf{k_2})\tilde{\theta}^{(1)}(\mathbf{k_3})\tilde{\theta}^{(\mathbf{3})}(\mathbf{k_4})\right\rangle$$

Connected Moment of $P_{q,perp}$ Next-to-leading order

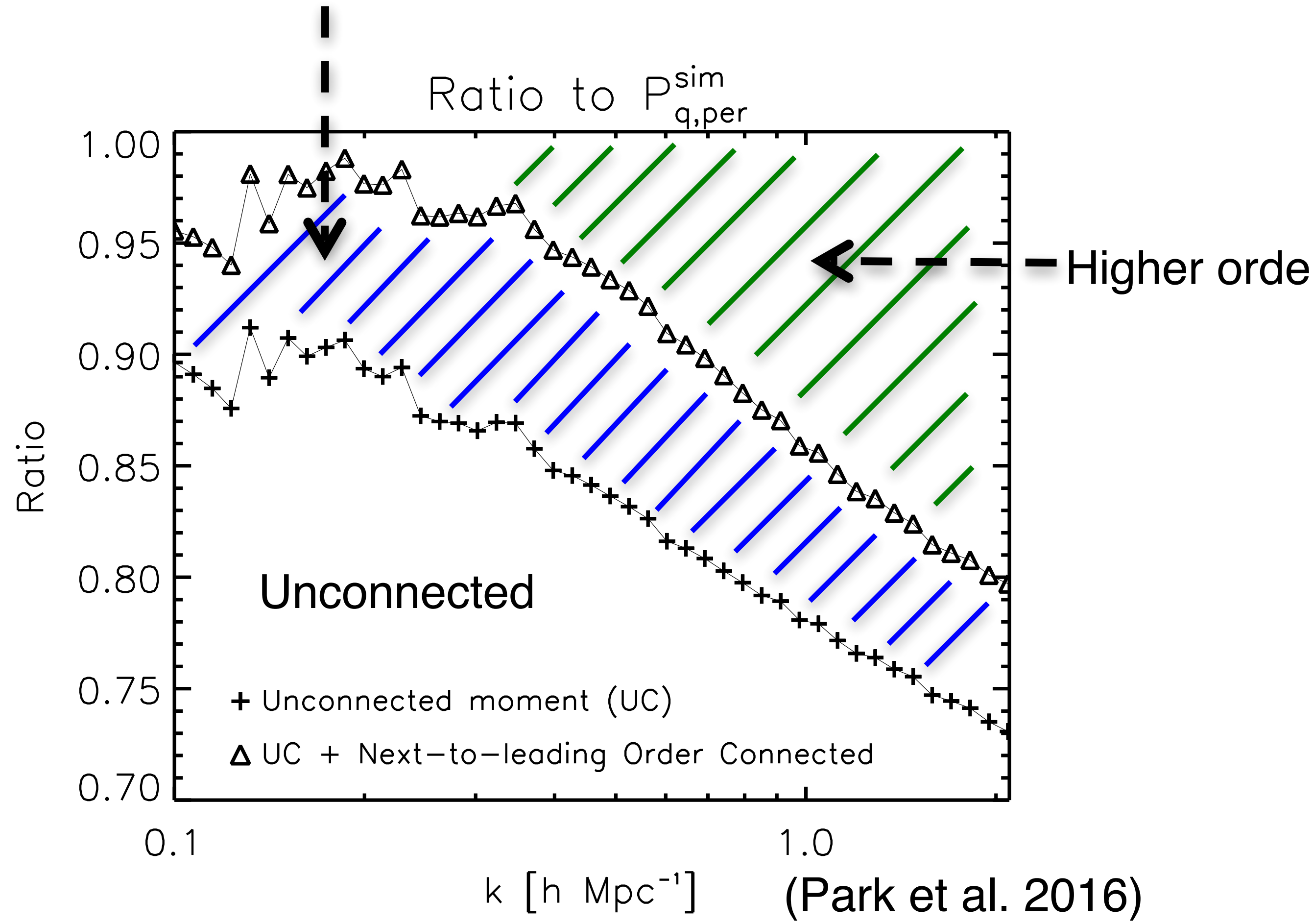
$$P_{q_\perp,c} = -\int \frac{d^3k'}{(2\pi)^3}\int \frac{d^3k''}{(2\pi)^3}\left[\frac{1-(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}')(\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}'')}{k'k''}\right]$$

$$\begin{aligned}
&[6[F_3^{(s)}(\mathbf{k}+\mathbf{k}'',-\mathbf{k}',-\mathbf{k}'')P(\mathbf{k}+\mathbf{k}'')P(\mathbf{k}')P(\mathbf{k}'')\\
&+F_3^{(s)}(\mathbf{k}-\mathbf{k}',\mathbf{k}',\mathbf{k}'')P(\mathbf{k}-\mathbf{k}')P(\mathbf{k}')P(\mathbf{k}'')\\
&+G_3^{(s)}(\mathbf{k}-\mathbf{k}',-\mathbf{k}-\mathbf{k}'',\mathbf{k}'')P(\mathbf{k}-\mathbf{k}')P(\mathbf{k}+\mathbf{k}'')P(\mathbf{k}'')\\
&+G_3^{(s)}(\mathbf{k}-\mathbf{k}',-\mathbf{k}-\mathbf{k}'',\mathbf{k}')P(\mathbf{k}-\mathbf{k}')P(\mathbf{k}+\mathbf{k}'')P(\mathbf{k}')]\\
\\
&+4[F_2^{(s)}(\mathbf{k}-\mathbf{k}',\mathbf{k}'+\mathbf{k}'')G_2^{(s)}(\mathbf{k}'+\mathbf{k}'',-\mathbf{k}'')P(\mathbf{k}-\mathbf{k}')P(\mathbf{k}'+\mathbf{k}'')P(\mathbf{k}'')\\
&+F_2^{(s)}(\mathbf{k}-\mathbf{k}',\mathbf{k}'+\mathbf{k}'')G_2^{(s)}(-\mathbf{k}'-\mathbf{k}'',\mathbf{k}')P(\mathbf{k}-\mathbf{k}')P(\mathbf{k}'+\mathbf{k}'')P(\mathbf{k}')\\
&+G_2^{(s)}(\mathbf{k},-\mathbf{k}+\mathbf{k}')F_2^{(s)}(\mathbf{k},\mathbf{k}'')P(\mathbf{k}-\mathbf{k}')P(\mathbf{k})P(\mathbf{k}'')\\
&+G_2^{(s)}(\mathbf{k},-\mathbf{k}+\mathbf{k}')G_2^{(s)}(\mathbf{k},-\mathbf{k}-\mathbf{k}'')P(\mathbf{k}-\mathbf{k}')P(\mathbf{k})P(\mathbf{k}+\mathbf{k}'')\\
&+G_2^{(s)}(\mathbf{k}-\mathbf{k}'+\mathbf{k}'',-\mathbf{k}+\mathbf{k}')F_2^{(s)}(\mathbf{k}-\mathbf{k}'+\mathbf{k}'',\mathbf{k}')P(\mathbf{k}-\mathbf{k}')P(\mathbf{k}-\mathbf{k}'+\mathbf{k}'')P(\mathbf{k}')\\
&+G_2^{(s)}(\mathbf{k}-\mathbf{k}'+\mathbf{k}'',-\mathbf{k}+\mathbf{k}')G_2^{(s)}(\mathbf{k}-\mathbf{k}'+\mathbf{k}'',-\mathbf{k}-\mathbf{k}'')P(\mathbf{k}-\mathbf{k}')P(\mathbf{k}-\mathbf{k}'+\mathbf{k}'')P(\mathbf{k}+\mathbf{k}'')\\
&+F_2^{(s)}(-\mathbf{k}'-\mathbf{k}'',\mathbf{k}+\mathbf{k}'')G_2^{(s)}(-\mathbf{k}'-\mathbf{k}'',\mathbf{k}'')P(\mathbf{k}+\mathbf{k}'')P(\mathbf{k}'+\mathbf{k}'')P(\mathbf{k}'')\\
&+F_2^{(s)}(-\mathbf{k}'-\mathbf{k}'',\mathbf{k}+\mathbf{k}'')G_2^{(s)}(-\mathbf{k}'-\mathbf{k}'',\mathbf{k}')P(\mathbf{k}+\mathbf{k}'')P(\mathbf{k}'+\mathbf{k}'')P(\mathbf{k}')\\
&+G_2^{(s)}(-\mathbf{k}-\mathbf{k}''+\mathbf{k}',\mathbf{k}+\mathbf{k}'')F_2^{(s)}(-\mathbf{k}-\mathbf{k}''+\mathbf{k}',\mathbf{k}'')P(\mathbf{k}+\mathbf{k}'')P(\mathbf{k}-\mathbf{k}'+\mathbf{k}'')P(\mathbf{k}'')\\
&+G_2^{(s)}(-\mathbf{k},\mathbf{k}+\mathbf{k}'')F_2^{(s)}(-\mathbf{k},\mathbf{k}')P(\mathbf{k}+\mathbf{k}'')P(\mathbf{k})P(\mathbf{k}')\\
&+F_2^{(s)}(\mathbf{k},-\mathbf{k}')F_2^{(s)}(\mathbf{k},\mathbf{k}'')P(\mathbf{k})P(\mathbf{k}')P(\mathbf{k}'')\\
&+F_2^{(s)}(\mathbf{k}-\mathbf{k}'+\mathbf{k}'',\mathbf{k}')F_2^{(s)}(\mathbf{k}-\mathbf{k}'+\mathbf{k}'',-\mathbf{k}'')P(\mathbf{k}')P(\mathbf{k}-\mathbf{k}'+\mathbf{k}'')P(\mathbf{k}'')]]
\end{aligned}$$

We have derived the connected term in the next-to-leading order.

(Park et al. 2016)

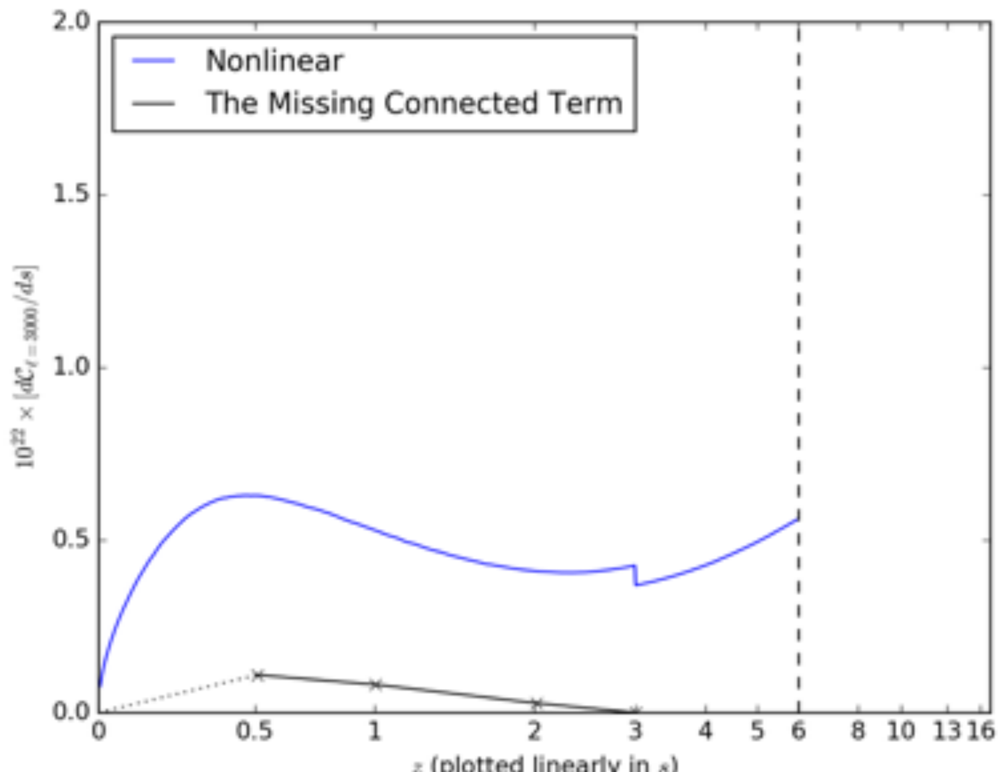
Connected Moment of $P_{q,perp}$
Next-to-leading order



(Park et al. 2016)

Next-to-leading order explains the deficit up to k ~ 0.3 *h*/Mpc !
At k > 0.3 h/Mpc, perhaps higher-order terms dominates.

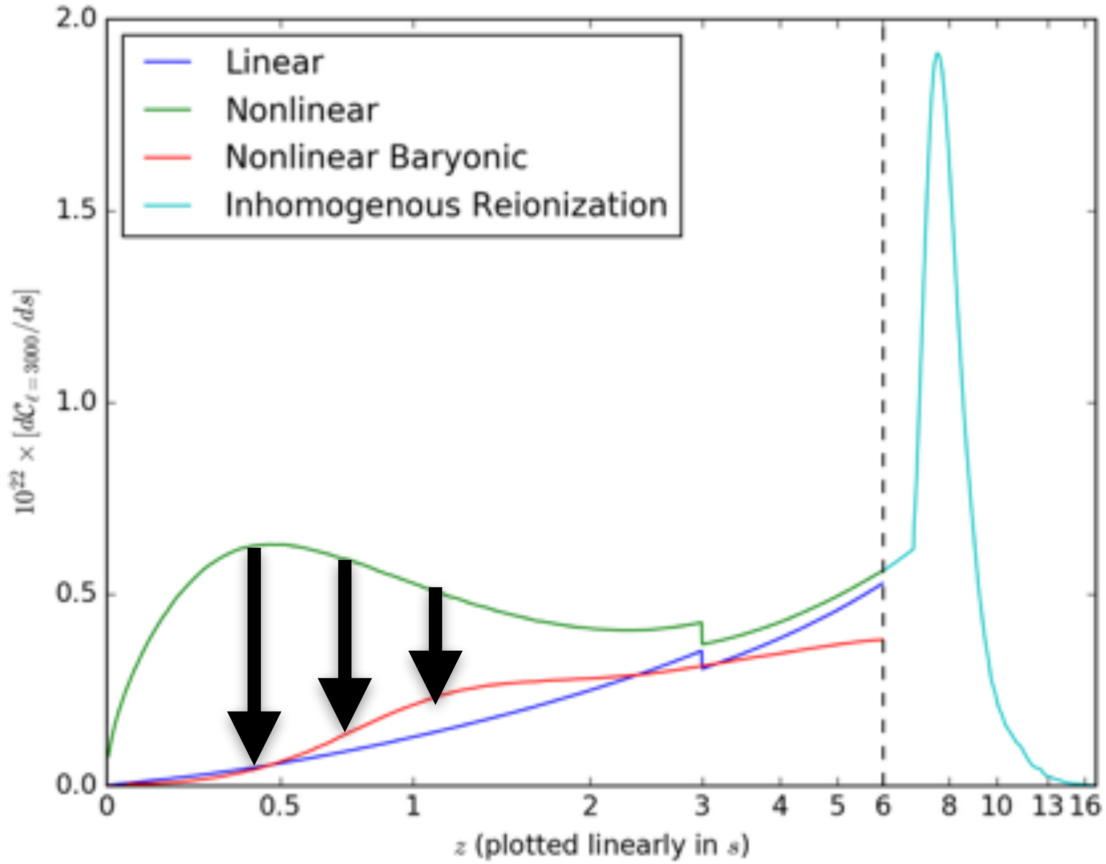
Final Result of Part 2



(Park et al. 2016)

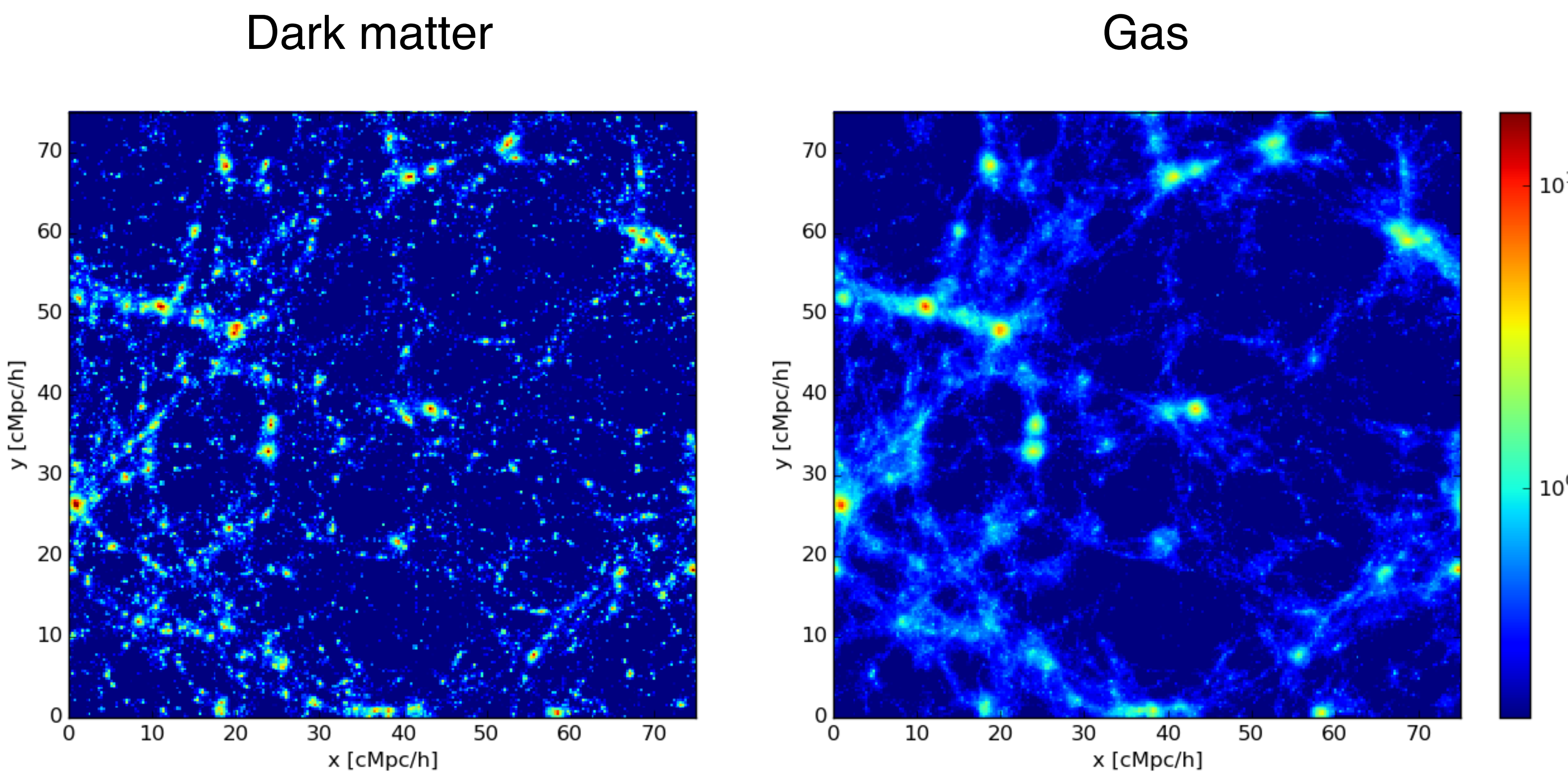
The connected term add ~10% extra to the post-reionization kSZ signal compared to the previous expression.

Part 3) Effect of Baryonic Physics in the kSZ signal



At low redshift, offset between dark matter and free electron densities due to baryonic physics becomes an important factor.

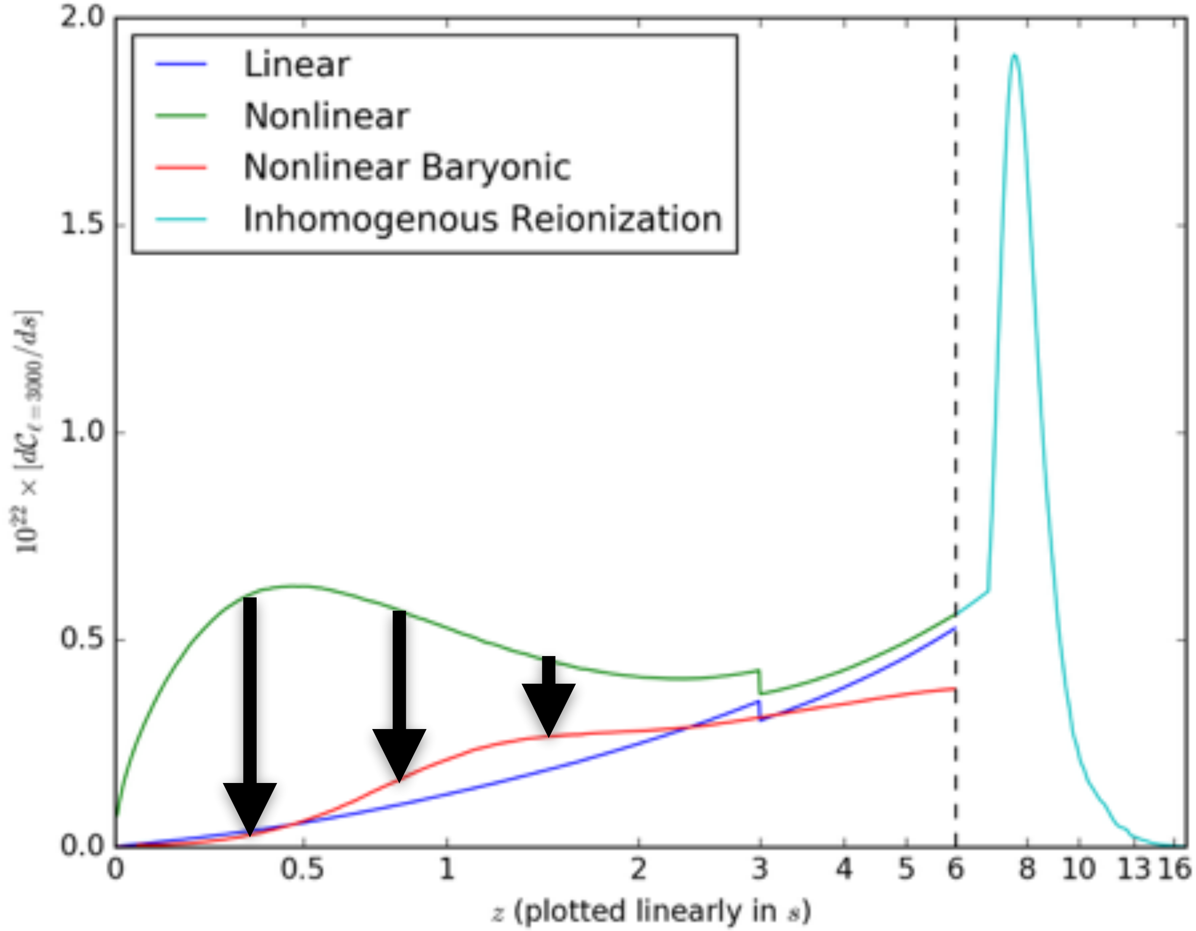
Part 3) Impact of Baryonic Physics on the kSZ signal



Two things happen to the baryonic density field at low z:

1) **Star formation locks some gas into stars.**
2) **Star formation feedback wipes out density structures in gas.**

Part 3) Effect of Baryonic Physics in the kSZ signal



$$\mathbf{q} = X(1+\delta)\mathbf{v} = (1+\delta_{nl,e})\mathbf{v}_{nl,e}$$

Both effects act toward suppressing the signal.

How to model the baryonic effect in the kSZ signal

The first step is to use the electron density power spectrum in the equation below.

$$P_{q_\perp} = \int \frac{d^3k'}{(2\pi)^3} \left(1 - \mu'^2\right)$$

$$\left[P_{\delta\delta}(|\mathbf{k}-\mathbf{k}'|)P_{vv}(\mathbf{k}') - \frac{k'}{|\mathbf{k}-\mathbf{k}'|}P_{\delta v}(|\mathbf{k}-\mathbf{k}'|)P_{\delta v}(\mathbf{k}')\right]$$

Assume the baryonic density field is related to the dark matter density field by a window function, *W*.

$$\tilde{\delta}_b(k) = W(k)\tilde{\delta}_{dm}(k) \quad \longrightarrow \quad \begin{aligned} P^b_{\delta\delta}(k) &= W^2(k)P^{dm}_{\delta\delta}(k) \\ P^b_{\delta v}(k) &= W(k)P^{dm}_{\delta v}(k) \end{aligned}$$

Illustris Simulation

Physics included
(in addition to gravity)

1. Gas Cooling
2. Star Formation
3. Stellar Evolution
4. Stellar Radiation Feedback
5. Supernova Feedback
6. Blackhole Feedback

Simulation Name	$L_{box}[Mpc]$	N_{DM}	m_{DM} $[M_\odot]$	m_{gas} $[M_\odot]$
Illustris-1	106.5	1820^3	6.3×10^6	1.3×10^6
Illustris-1-Dark	106.5	1820^3	7.5×10^6	0
Illustris-2	106.5	910^3	5.0×10^7	1.0×10^7
Illustris-2-Dark	106.5	910^3	6.0×10^7	0
Illustris-3	106.5	455^3	4.0×10^8	8.1×10^7
Illustris-3-Dark	106.5	455^3	4.8×10^8	0

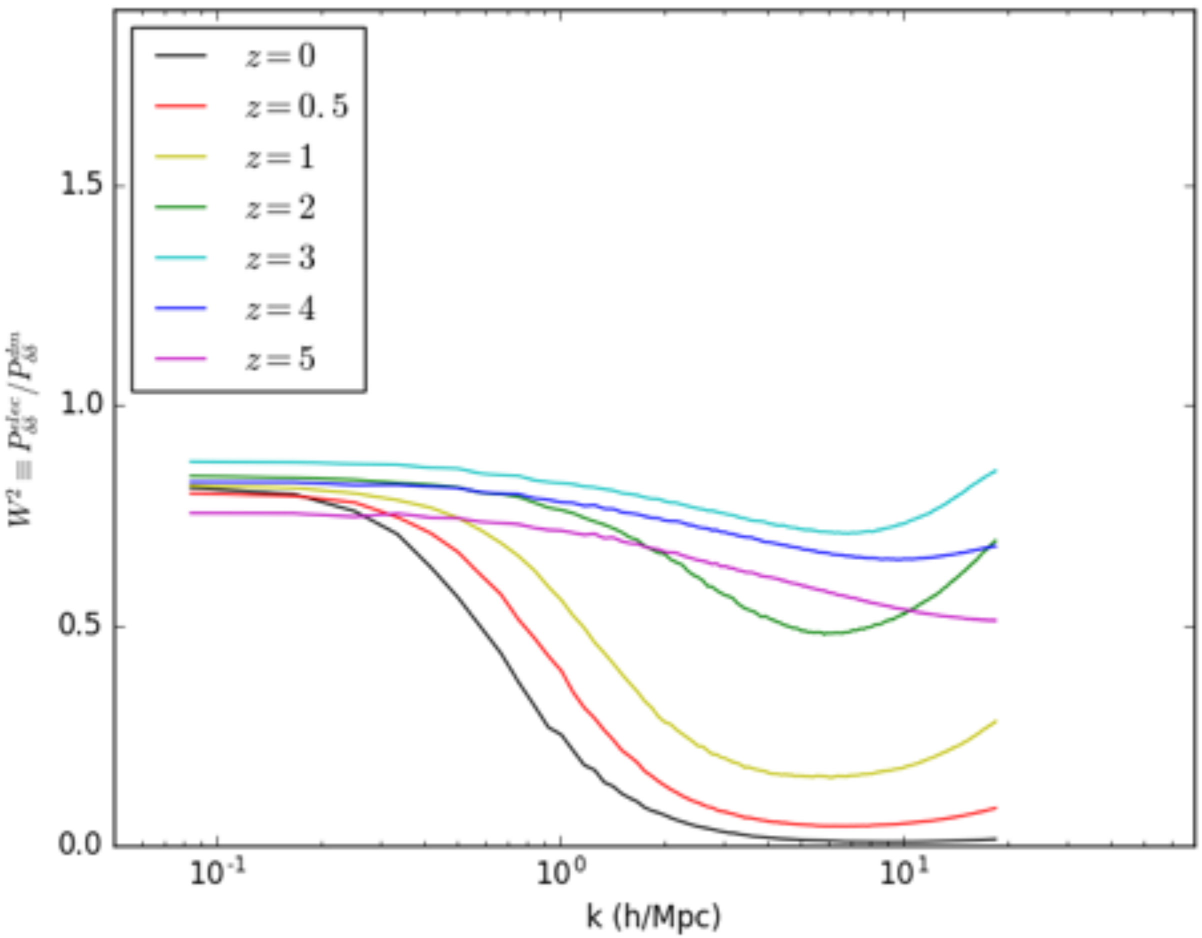
Illustris is a high resolution simulation that includes a list of crucial baryonic physics that allows to create present day galaxies.

Illustris Simulation

Video from http://www.illustris-project.org/media/

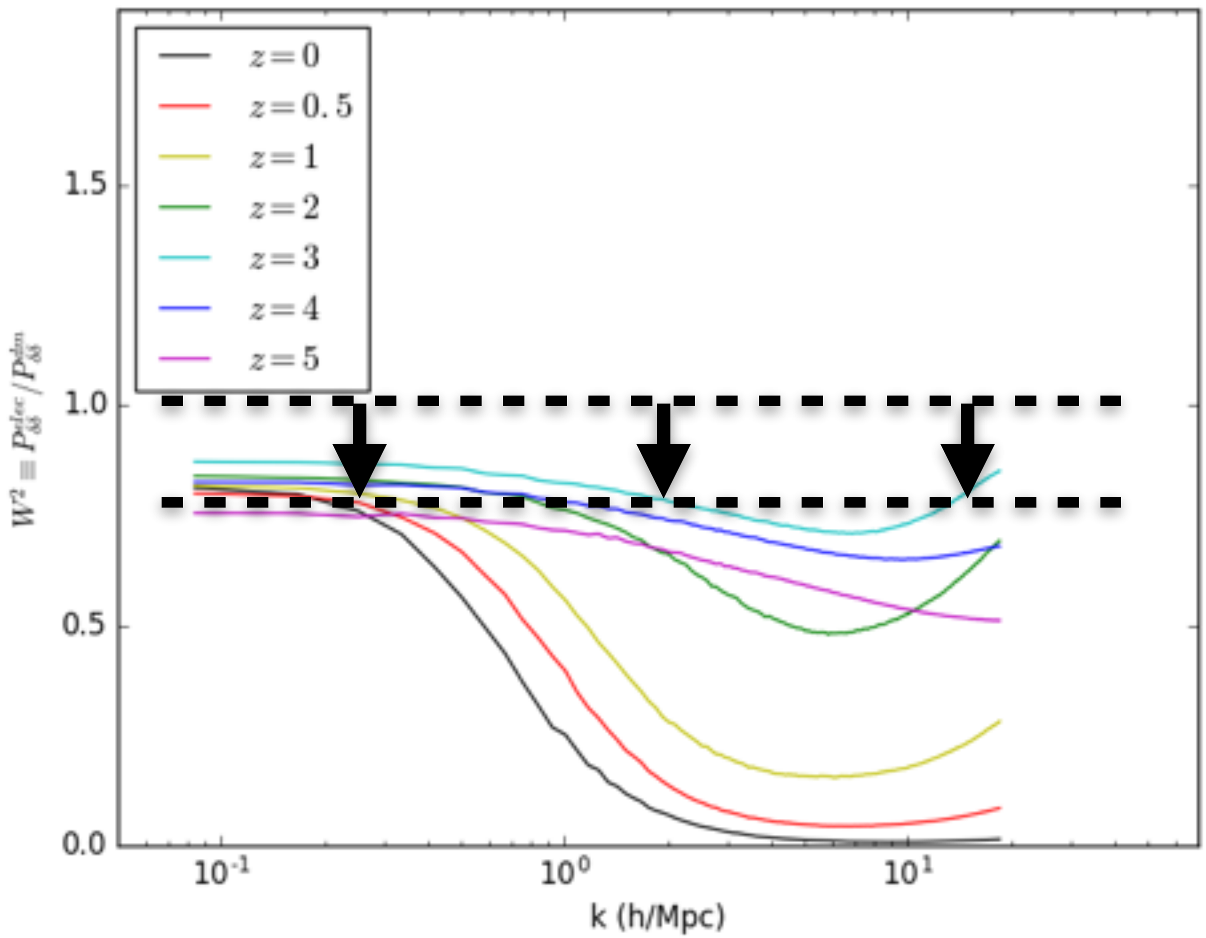
Illustris has a fairly large size (75 Mpc/*h*) with baryonic physics (star-formation & feedback) needed for kSZ calculation.

Window function from Illustris



(Park et al. in prep.)

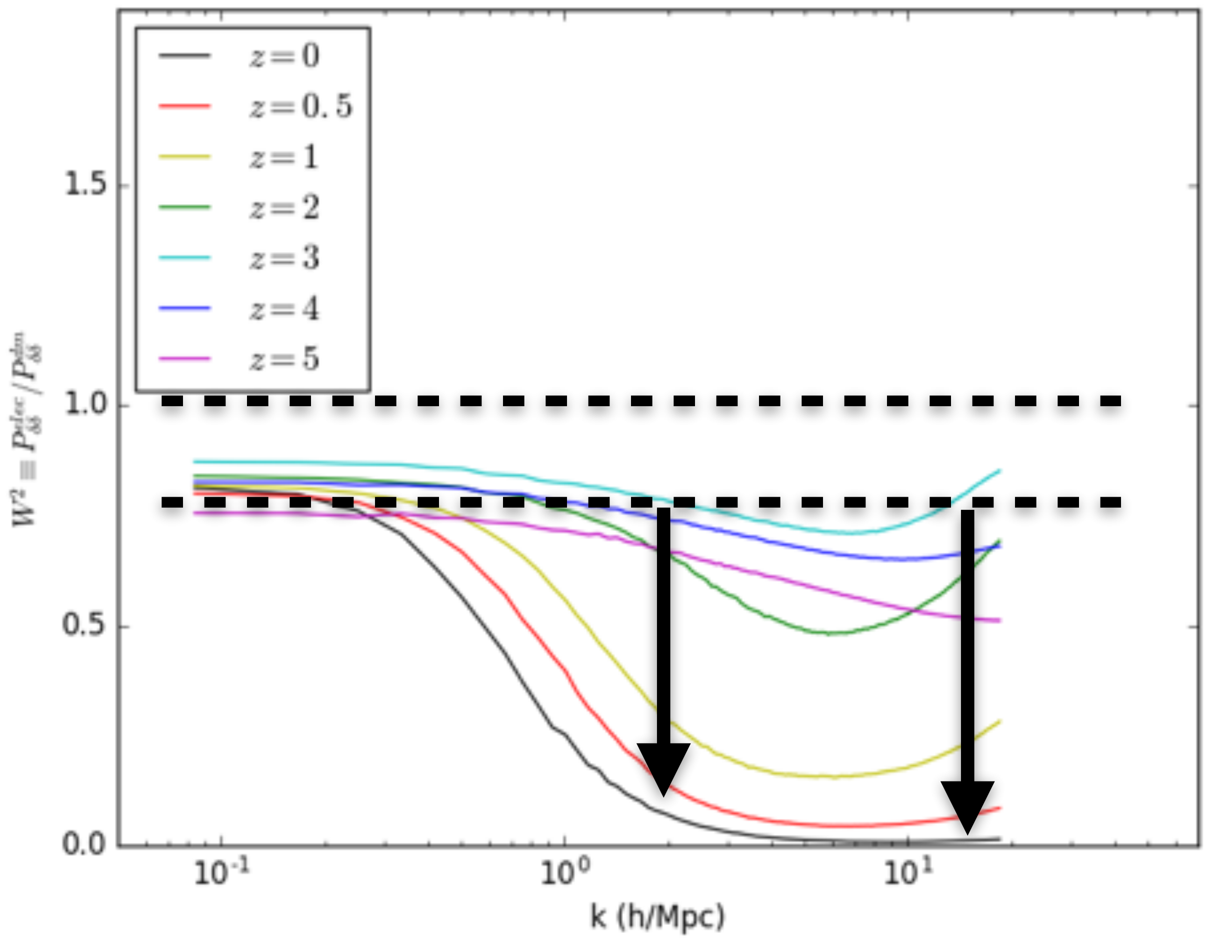
Window function from Illustris



Suppression at all scales due to gas locked into stellar products.

(Park et al. in prep.)

Window function from Illustris

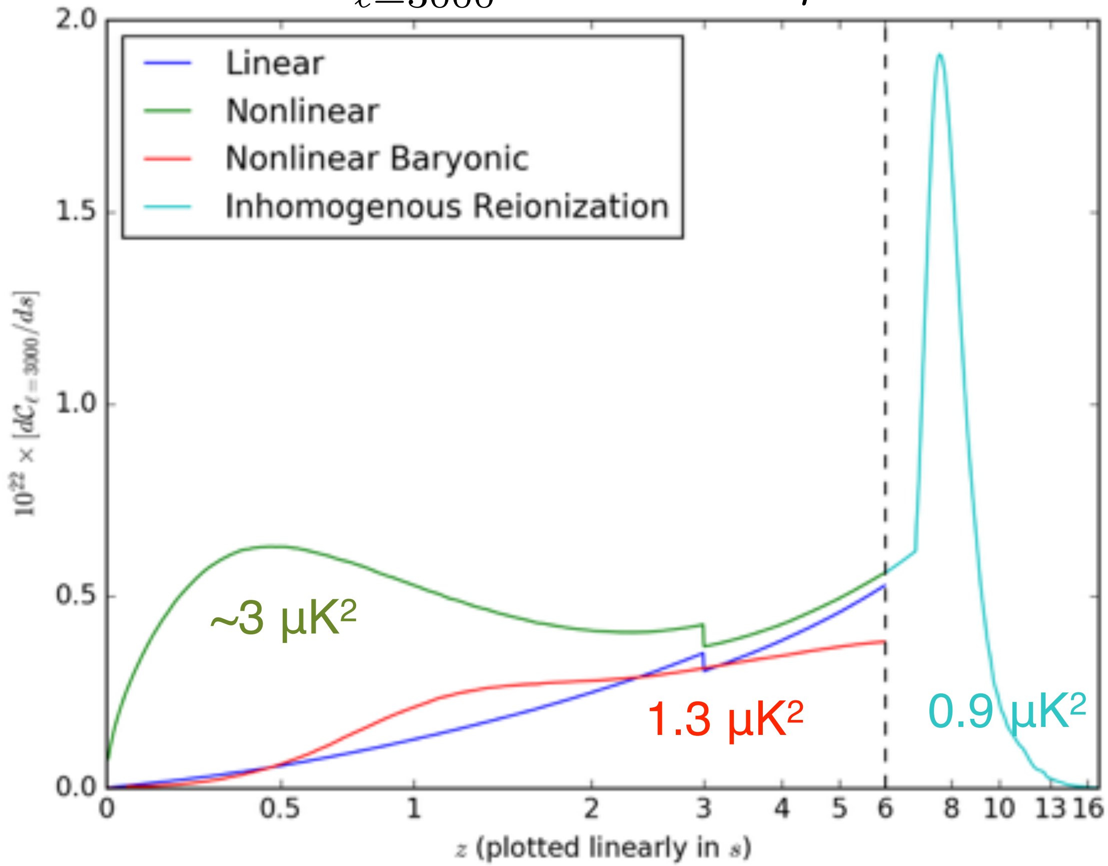


Extra suppression at small scales and low-redshift due to pressure.

(Park et al. in prep.)

Result for Part 3
Post-reionization kSZ signal

$$\mathcal{D}_{\ell=3000}^{\rm SPT} = 2.9 \pm 1.3\ \mu K^2$$



We find 1.3 μK² for the post-reionization signal accounting for both the effects of nonlinear growth of structures and baryonic physics.

(Park et al. in prep.)

Summary

- The impact of low-mass galaxies are not distinguishable in the kSZ signal at $l = 3000$ where we currently observe, but it is at $l = 10000$.
- There is a missing term in the expression for $P_{q,perp}$ from previous literature. That term adds ~10% extra to the post-reionization signal.
- Using Illustris simulation we model baryonic effects in the post-reionization ($z < 6$) kSZ signal and include it in the calculation. The result is $D_{l=3000} = 1.3\ \mu K^2$.