A Relativistic Magnetohydrodynamic (RMHD) Code based on an Upwind scheme

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1. Introduction

Upwind Schemes

Upwind schemes solve hyperbolic partial differential equations by simulating direction of propagation of information in a flow field

➡ Eigenvalues and eigenvectors of the system are required

➡ But they have not yet been analytically given for RMHDs

Previous studies Using fully upwind schemes

Balsara (2001) – numerical calculations of eigenvalues and eigenvectors

Anton et al (2010) – analytic formulae for eigenvectors, but numerical calculations of eigenvalues

Most of recent codes for RMHDs are based on HLL, HLLC, HLLD...

2. Conservation Equations

Conservation equations

$$\frac{\partial \vec{q}}{\partial t} + \frac{\partial \overrightarrow{F_j}}{\partial x_j} = 0$$

$$\vec{q} = \begin{bmatrix} D \\ M_i \\ E \\ B_y \\ B_z \end{bmatrix}$$

D → Mass density

M_i → Momentum densities

E → Energy density

state vector

$$\overrightarrow{F_j} = \begin{bmatrix} D v_j \\ M_i v_j - B_j[B_i/\Gamma^2 + v_i(\vec{v} \cdot \vec{B})] + \delta_{ij}(p_g - p_m + B^2) \\ (E + p_g + p_m) v_j - B_i(\vec{v} \cdot \vec{B}) \\ B_y v_x - B_x v_y \\ B_z v_x - B_x v_z \end{bmatrix}$$

flux vector

ρ : proper mass density

p_g : gas pressure

v_i : fluid velocity

h : specific enthalpy

B_i : Magnetic field

$$D = \rho \Gamma$$

$$M_i = \rho h \Gamma^2 v_i + B^2 v_i - B_i(\vec{v} \cdot \vec{B})$$

$$E = \rho h \Gamma^2 - (p_g + p_m) + B^2$$

$$\Gamma = \frac{1}{\sqrt{1 - v^2}}$$

Lorentz factor (c = 1)

magnetic pressure

$$p_m = \frac{1}{2}[B^2/\Gamma^2 + (\vec{v} \cdot \vec{B})]$$

2. Conservation Equations

Jacobian matrix A_j

$$\frac{\partial \vec{q}}{\partial t} + \frac{\partial \overrightarrow{F_j}}{\partial x_j} = \frac{\partial \vec{q}}{\partial t} + A_j \frac{\partial \vec{q}}{\partial x_j} = 0$$

$$A_j = \frac{\partial \overrightarrow{F_j}}{\partial \vec{q}} = \frac{\partial \overrightarrow{F_j}}{\partial \vec{u}} \frac{\partial \vec{u}}{\partial \vec{q}}$$

$$A_x = \frac{1}{N} \begin{pmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} & A_{27} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} & A_{36} & A_{37} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} & A_{46} & A_{37} \\ N & 0 & 0 & 0 & 0 & 0 & 0 \\ A_{61} & A_{62} & A_{63} & A_{64} & A_{65} & A_{66} & A_{67} \\ A_{71} & A_{72} & A_{73} & A_{74} & A_{75} & A_{76} & A_{77} \end{pmatrix}$$

$$\vec{u} = \begin{bmatrix} \rho \\ v_x \\ v_y \\ v_z \\ p \\ B_y \\ B_z \end{bmatrix}$$

parameter vector

$$N = hn[b^2/\Gamma^2 + (\vec{v} \cdot \vec{b})^2(1 - c_s^2) + \Gamma^2(1 - c_s^2 v^2)]$$

$n = \rho \frac{\partial h}{\partial p} - 1$: polytropic index

$c_s^2 = -\frac{\rho}{nh}\frac{\partial h}{\partial \rho}$: sound speed

$$b_i = \frac{B_i}{\sqrt{\rho h}}$$

2. Conservation Equations

Jacobian matrix A_j

$$\begin{aligned}
A_{11} = {} & hb_x^2 v_x[1 - v^2 + n(1-c_s^2)(1-v_y^2-v_z^2)] + hnb_y^2 v_x(1 - v_x^2 - c_s^2 v_y^2 - v_z^2) \\
& + hnb_z^2 v_x(1 - v_x^2 - v_y^2 - c_s^2 v_z^2) + hb_x(b_y v_y + b_z v_z)[2n(1-c_s^2)v_x^2 + (1-nc_s^2)(1-v^2)] \\
& + 2hnb_y b_z v_x v_y v_z(1-c_s^2) + hv_x[1 + n(1-c_s^2)\Gamma^2]
\end{aligned}$$

$$\begin{aligned}
A_{12} = {} & b_x^2 n(1-v^2) - n(\vec{v}\times\vec{b}|_x)^2(1-c_s^2) + b_x v_x(\vec{v}\cdot\vec{b})(1+n) \\
& + \Gamma^2[n(1-c_s^2 v^2) + (1+nc_s^2)v_x^2] + b^2 n(\vec{v}\times\vec{b}|_x)^2(1-c_s^2)/(\Gamma^2+b^2)
\end{aligned}$$

$$\begin{aligned}
A_{13} = {} & b_x^2 v_x v_y(1+n) + b_z v_x n(\vec{v}\times\vec{b}|_x)(1-c_s^2) \\
& + b_x b_y[v_y^2 + n(1-v_x^2) - nc_s^2 v_z^2] + b_x b_z v_y v_z(1+nc_s^2) + \Gamma^2 v_x v_y(1+nc_s^2) \\
& + b^2 n(\vec{v}\times\vec{b}|_x)(\vec{v}\times\vec{b}|_y)(1-c_s^2)/(\Gamma^2+b^2)
\end{aligned}$$

$$\begin{aligned}
A_{14} = {} & b_x^2 v_x v_z(1+n) - b_y v_x n(\vec{v}\times\vec{b}|_x)(1-c_s^2) \\
& + b_x b_y v_y v_z(1+nc_s^2) + b_z b_x[v_z^2 + n(1-v_x^2-c_s^2 v_y^2)] \\
& + \Gamma^2 v_x v_z(1+nc_s^2) + b^2 n(\vec{v}\times\vec{b}|_x)(\vec{v}\times\vec{b}|_z)(1-c_s^2)/(\Gamma^2+b^2)
\end{aligned}$$

$$A_{15} = -(1+n)[b_x(\vec{v}\cdot\vec{b}) + \Gamma^2 v_x]$$

$$\begin{aligned}
A_{16} = {} & b_x^3 v_y[v_x^2 + n(1-v_y^2-v_z^2)] + 2b_x^2 b_z v_x v_y v_z(1+n) \\
& + b_x^2 b_y v_x(1 - v_x^2 + v_y^2 - v_z^2 + 2nv_y^2) + b_y^2 b_x v_y(1+n)(1-v_x^2-v_z^2) \\
& + b_z^2 b_x v_y[v_z^2 + n(1-v_x^2-v_y^2)] + b_x b_y b_z v_z(1 - v_x^2 + v_y^2 - v_z^2 + 2nv_y^2) \\
& + b_x\Gamma^2 v_y[v_x^2 + n(1-v_x^2) + nc_s^2(v_x^2-v_y^2-v_z^2)] + b_y\Gamma^2 v_x[(1-n)(1-v_x^2-v_z^2) + 2nc_s^2 v_y^2] \\
& + b_z\Gamma^2 v_x v_y v_z(1-n+2nc_s^2) - \Gamma^2 n(\vec{v}\cdot\vec{b})(\vec{v}\times\vec{b}|_x)(\vec{v}\times\vec{b}|_y)(1-c_s^2)/(\Gamma^2+b^2)
\end{aligned}$$

$$\begin{aligned}
A_{17} = {} & b_x^3 v_z[v_x^2 + n(1-v_z^2-v_y^2)] + b_x^2 b_z v_x[1 - v_x^2 - v_y^2 + (1+2n)v_z^2] \\
& + 2b_x^2 b_y v_x v_y v_z(1+n) + b_y^2 b_x v_z[v_y^2 + n(1-v_x^2-v_z^2)] + b_z^2 b_x v_z(1+n)(1-v_y^2-v_x^2) \\
& + b_x b_y b_z v_y(1 - v_x^2 - v_y^2 + (1+2n)v_z^2) + b_x\Gamma^2 v_z[v_x^2 + n(1-v_x^2) + nc_s^2(v_x^2-v_y^2-v_z^2)] \\
& + b_y\Gamma^2 v_x v_y v_z[1 - n(1-2c_s^2)] + b_z\Gamma^2 v_x[(1-n)(1-v_x^2-v_y^2) + 2nc_s^2 v_z^2] \\
& - \Gamma^2 n(\vec{v}\cdot\vec{b})(\vec{v}\times\vec{b}|_x)(\vec{v}\times\vec{b}|_z)(1-c_s^2)/(\Gamma^2+b^2)
\end{aligned}$$

$$A_{21} = -h^2(1-nc_s^2)[b^2 - b_x^2 v^2 - (\vec{v}\times\vec{b}|_x)^2 - (\vec{v}\times\vec{b}|_y)^2 + \Gamma^2(1-v_x^2)]$$

$$\begin{aligned}
A_{22} = {} & -hb_x^2 v_x[(1+nc_s^2)(1-v_x^2-2v_y^2-2v_z^2) - 2n(1-c_s^2 v_x^2-v_y^2-v_z^2)] \\
& - hb_y^2 v_x[(1-n)(1-v_x^2) + 2n(c_s^2 v_y^2 + v_z^2)] - hb_z^2 v_x[(1-n)(1-v_x^2) + 2n(v_y^2 + c_s^2 v_z^2)] \\
& - hb_x b_y v_y[2v_x^2 + n(1-c_s^2)(1-3v_x^2) + 2nc_s^2(1-v_y^2-v_z^2)] + 4hnb_y b_z v_x v_y v_z(1-c_s^2) \\
& - hb_z b_x v_z[2v_x^2 + n(1-c_s^2)(1-3v_x^2) + 2nc_s^2(1-v_y^2-v_z^2)] \\
& + h\Gamma^2 v_x[2n - (1-nc_s^2)(1-v_x^2) - 2c_s^2 n(v_z^2+v_y^2+1)] \\
& + 2hnb_x(\vec{v}\cdot\vec{b})(\vec{v}\times\vec{b}|_x)^2(1-c_s^2)/(\Gamma^2+b^2)
\end{aligned}$$

$$\begin{aligned}
A_{23} = {} & hb_x^2 v_y[2(v_y^2+v_z^2) - (1+n)(1-v_x^2) + 2n(1-c_s^2 v_x^2)] - hb_y^2 v_y(1+nc_s^2)(1-v_x^2) \\
& - hb_z^2 v_y(1+n)(1-v_x^2) - hb_x b_y v_x[2(1+nc_s^2)v_y^2 - n(1-c_s^2)(1-v_x^2)] \\
& + hnb_y b_z v_z(1-c_s^2)(1-v_x^2) - 2hb_z b_x v_x v_y v_z(1+nc_s^2) - h\Gamma^2 v_y(1+nc_s^2)(1-v_x^2) \\
& + 2hnb_x(\vec{v}\cdot\vec{b})(\vec{v}\times\vec{b}|_x)(\vec{v}\times\vec{b}|_y)(1-c_s^2)/(\Gamma^2+b^2)
\end{aligned}$$

$$\begin{aligned}
A_{24} = {} & hb_x^2 v_z[2(v_y^2+v_z^2) - (1+n)(1-v_x^2) + 2n(1-c_s^2 v_x^2)] - hb_y^2 v_z(1+n)(1-v_x^2) \\
& - hb_z^2 v_z(1+nc_s^2)(1-v_x^2) - 2hb_x b_y v_x v_y v_z(1+nc_s^2) + hnb_y b_z v_y(1-c_s^2)(1-v_x^2) \\
& - hb_z b_x v_x[2(1+nc_s^2)v_z^2 - n(1-c_s^2)(1-v_x^2)] - h\Gamma^2 v_z(1+nc_s^2)(1-v_x^2) \\
& + 2hnb_x(\vec{v}\cdot\vec{b})(\vec{v}\times\vec{b}|_x)(\vec{v}\times\vec{b}|_z)(1-c_s^2)/(\Gamma^2+b^2)
\end{aligned}$$

$$\begin{aligned}
A_{25} = {} & hb_x^2[1 - (2+n)v^2 + (1+nc_s^2)v_x^2] + hb_z^2[1 - v_x^2 + n(v_y^2 + c_s^2 v_z^2)] \\
& + hb_y^2[1 - v_x^2 + n(c_s^2 v_y^2 + v_z^2)] + 2hb_x v_x(b_y v_y + b_z v_z)(1+nc_s^2) \\
& - 2hnb_y b_z v_y v_z(1-c_s^2) + h\Gamma^2[(1+n)(1-v_x^2) - n(1-c_s^2 v^2)]
\end{aligned}$$

$$\begin{aligned}
A_{26} = {} & hb_x^3 v_x v_y[2n(1-c_s^2 v_x^2-v_z^2-v_y^2) - (1+nc_s^2)(1-v_x^2-2v_y^2-2v_z^2)] \\
& - hb_z^3 v_y v_z(1+nc_s^2)(1-v_x^2) - hb_y^3(1-v_x^2)[(1-n)(1-v_x^2-v_z^2) + 2nc_s^2 v_y^2] \\
& - hb_x^2 b_y[(1-v_x^2-v_z^2)(1-v_x^2-2v_z^2) - 2v_y^2(1-2v_x^2-v_z^2) - n(1-v_y^2-v_z^2)(1-2v_z^2) \\
& \qquad + nv_x^2(1-5v_y^2-3v_z^2) + nc_s^2 v_y^2(1-2v_y^2-2v_z^2) + c_s^2 n v_x^2(1-v_x^2+5v_y^2)] \\
& - hb_x^2 b_z v_y v_z[(1-n+nc_s^2)(1+v_x^2-2v_y^2-2v_z^2) - n - n(1-4c_s^2)v_x^2] \\
& + hb_y^2 b_x v_x v_y[2(1-n)v_z^2 + 4n(1-c_s^2 v_y^2-v_z^2) - 3(1+nc_s^2)(1-v_x^2)] \\
& - hb_y^2 b_z v_y v_z(1-v_x^2)[1-n(2-3c_s^2)] - hb_z^2 b_x v_x v_y[(1+nc_s^2)(1-v_x^2+2v_z^2) - 2n(1-c_s^2)v_z^2] \\
& - hb_y b_z^2(1-v_x^2)[1 - v_x^2 - v_z^2 - n(1-v_x^2-v_y^2) + nc_s^2(v_y^2+v_z^2)] \\
& - 2hb_x b_y b_z v_x v_z[(1-n)(v_y^2-v_z^2) + (1-2n+nc_s^2)(1-v_x^2) + 4c_s^2 n v_y^2] \\
& - h\Gamma^2 v_y(b_x v_x + b_z v_z)(1-v_x^2)[1-n(1-2c_s^2)] \\
& - hb_y\Gamma^2(1-v_x^2)[(1-n)(1-v_x^2-v_z^2) + 2nc_s^2 v_y^2] \\
& + 2hnb_x(\vec{v}\cdot\vec{b})^2(\vec{v}\times\vec{b}|_x)(\vec{v}\times\vec{b}|_y)(1-c_s^2)/(\Gamma^2+b^2)
\end{aligned}$$

$$\begin{aligned}
A_{27} = {} & -hb_x^3 v_x v_z[(1+nc_s^2)(1-2v_z^2-2v_y^2-v_x^2) - 2n(1-c_s^2 v_z^2-v_y^2-v_x^2)] \\
& - hb_y^3 v_y v_z(1+nc_s^2)(1-v_x^2) - hb_z^3(1-v_x^2)[(1-n)(1-v_y^2-v_x^2) + 2nc_s^2 v_z^2] \\
& - hb_x^2 b_y v_y v_z[(1-n+nc_s^2)(1+v_x^2-2v_y^2-2v_z^2) - n - n(1-4c_s^2)v_x^2] \\
& + hb_x^2 b_z[2(1-2v_x^2-v_y^2)v_z^2 - (1-v_x^2-v_y^2)(1-v_x^2-2v_y^2) \\
& \qquad + n(1-2v_y^2)(1-v_y^2-c_s^2 v_z^2-v_x^2) - nv_x^2(1-5v_z^2-3v_y^2) \\
& \qquad + nc_s^2(v_x^2-v_z^2)(v_x^2-2v_z^2) - nc_s^2 v_x^2(2v_z^2+1)] \\
& - hb_y^2 b_x v_x v_z[(1+nc_s^2)(1-v_x^2+2v_y^2) - 2n(1-c_s^2)v_y^2] \\
& - hb_y^2 b_z(1-v_x^2)[1 - v_x^2 - v_y^2 - n(1-v_x^2-v_z^2) + nc_s^2(v_y^2+v_z^2)] \\
& + hb_z^2 b_x v_x v_z[2(1-n)v_y^2 + 4n(1-v_x^2-c_s^2 v_z^2) - 3(1+nc_s^2)(1-v_x^2)] \\
& - hb_z^2 b_y v_y v_z(1-v_x^2)[1-n(2-3c_s^2)] \\
& - 2hb_x b_y b_z v_x v_y[(1-2n+nc_s^2)(1-v_x^2) - (1-n)(v_y^2-v_z^2) + 4nc_s^2 v_z^2] \\
& - h\Gamma^2 v_z(b_y v_y + b_x v_x + b_z v_z)(1-v_x^2)[1-n(1-2c_s^2)] - b_z h(1-n)(1-v_x^2) \\
& + 2hnb_x(\vec{v}\cdot\vec{b})^2(\vec{v}\times\vec{b}|_x)(\vec{v}\times\vec{b}|_z)(1-c_s^2)/(\Gamma^2+b^2)
\end{aligned}$$

$\vdots$

3. Eigen-Structure

Analytic forms of Eigenvalues

$$\det(A_x - a_i) = \underbrace{(a_i - v_x)}_{\text{Entropy mode}}\underbrace{(A_2 a_i^2 + A_1 a_i + A_0)}_{\text{Alfven modes}}\underbrace{(C_4 a_i^4 + C_3 a_i^3 + C_2 a_i^2 + C_1 a_i + C_0)}_{\text{Compressible modes}}$$

Entropy mode

Alfven modes

Compressible modes

$$a_{entropy} = v_x$$

$$[(\Gamma^2 + b^2)(a_i - v_x) + b_x(\vec{v}\cdot\vec{b})]^2 - b_x^2[1 + b^2 - (\vec{v}\times\vec{b})^2] = 0$$

$$a_{Alfven}^{-} = \frac{v_x(\Gamma^2 + b^2) - b_x(\vec{v}\cdot\vec{b}) - |\, b_x \,| \sqrt{1 + b^2 - (\vec{v}\times\vec{b})^2}}{\Gamma^2 + b^2}$$

$$a_{Alfven}^{+} = \frac{v_x(\Gamma^2 + b^2) - b_x(\vec{v}\cdot\vec{b}) + |\, b_x \,| \sqrt{1 + b^2 - (\vec{v}\times\vec{b})^2}}{\Gamma^2 + b^2}$$

3. Eigen-Structure

Analytic forms of Eigenvalues

$$\det(A_x - a_i) = (a_i - v_x)(A_2 a_i^2 + A_1 a_i + A_0)(C_4 a_i^4 + C_3 a_i^3 + C_2 a_i^2 + C_1 a_i + C_0)$$

Entropy mode — $(a_i - v_x)$

Alfven modes — $(A_2 a_i^2 + A_1 a_i + A_0)$

Compressible modes — $(C_4 a_i^4 + C_3 a_i^3 + C_2 a_i^2 + C_1 a_i + C_0)$

$$\Gamma^2(1-c_s^2)(a_i-v_x)^4-(1-a_i^2)(a_i-v_x)^2[b^2+c_s^2-(\vec{v}\times\vec{b})^2]$$
$$+c_s^2(1-a_i^2)[b_x/\Gamma^2-(\vec{v}\cdot\vec{b})(a_i-v_x)]^2=0$$

General solutions for quartic equations are too complicated to use in the code.

Eigenvalues are correspond to characteristic speeds of the fluids, so they have to be real.

Using this criteria, we have obtained analytic forms for eigenvalues.

3. Eigen-Structure

Analytic forms of Eigenvalues

$$\det(A_x - a_i) = \underbrace{(a_i - v_x)}_{\text{Entropy mode}}\underbrace{(A_2 a_i^2 + A_1 a_i + A_0)}_{\text{Alfven modes}}\underbrace{(C_4 a_i^4 + C_3 a_i^3 + C_2 a_i^2 + C_1 a_i + C_0)}_{\text{Compressible modes}}$$

Entropy mode — Alfven modes — Compressible modes

$$\Gamma^2(1-c_s^2)(a_i-v_x)^4-(1-a_i^2)(a_i-v_x)^2[b^2+c_s^2-(\vec{v}\times\vec{b})^2]$$
$$+c_s^2(1-a_i^2)[b_x/\Gamma^2-(\vec{v}\cdot\vec{b})(a_i-v_x)]^2=0$$

$$\lambda_1 = -A-\sqrt{B+C} \qquad \lambda_2 = -A+\sqrt{B+C}$$
$$\lambda_3 = \quad A-\sqrt{B-C} \qquad \lambda_4 = \quad A+\sqrt{B-C}$$

$$a_1(^{-}_{fast}) = v_x+\lambda_1 \qquad a_2(^{-}_{slow}) = v_x+\lambda_2$$
$$a_3(^{+}_{slow}) = v_x+\lambda_3 \qquad a_4(^{+}_{fast}) = v_x+\lambda_4$$

$$a_1(^{-}_{fast}) \le a_2(^{-}_{Alfven}) \le a_3(^{-}_{slow}) \le a_4(=v_x) \le a_5(^{+}_{slow}) \le a_6(^{+}_{Alfven}) \le a_7(^{+}_{fast})$$

3. Eigen-Structure

Analytic forms of Eigenvectors

With the eigenvalues, we can express eivenvectors in a relatively simple form.

Right Eigenvectors

$$(A_x - a_i) \cdot \overrightarrow{R_i} = 0$$

$$\overrightarrow{R_i} = \left[R_{i1}\, ,\, R_{i2}\, ,\, R_{i3}\, ,\, R_{i4}\, ,\, R_{i5}\, ,\, R_{i6}\, ,\, R_{i7} \right]^T$$

Left Eigenvectors

$$\overrightarrow{L_i} \cdot (A_x - a_i) = 0$$

$$\overrightarrow{L_i} = \frac{1}{\widetilde{L}_i} \left[L_{i1}\, ,\, L_{i2}\, ,\, L_{i3}\, ,\, L_{i4}\, ,\, L_{i5}\, ,\, L_{i6}\, ,\, L_{i7} \right]$$

When some eigenvalues become same,
all components of the eigenvectors go to zero

<u>Singularity occurs!</u>

→ **Degeneracy issue**

3. Eigen-Structure

Degeneracy issue

$D_i = b_x/(a_i - v_x)$, $Y_{1i} = [a_i(\vec{v} \times \vec{b}|_z) - b_y]$, $Y_{2i} = -[a_i(\vec{v} \times \vec{b}|_y) + b_z]$, $Y_i = \sqrt{Y_{1i}^2 + Y_{2i}^2}$

$AA_i = (a_i - a^+_{Alfven})(a_i - a^-_{Alfven})(\Gamma^2 + b^2)$: Alfven mode equation

Degeneracy Case1 When bx = 0

$$a_1(^-_{fast}) \le a_2(^-_{Alfven}) \le a_3(^-_{slow}) \le a_4(=v_x) \le a_5(^+_{slow}) \le a_6(^+_{Alfven}) \le a_7(^+_{fast})$$

Degeneracy Case2 When AAi = 0, Yi=0

$$a_1(^-_{fast}) \le a_2(^-_{Alfven}) \le a_3(^-_{slow}) \le a_4(=v_x) \le a_5(^+_{slow}) \le a_6(^+_{Alfven}) \le a_7(^+_{fast})$$

Degeneracy Case3 When AAi = 0, Yi=0,

$$c_s^2 = [b^2/\Gamma^2 + (\vec{v} \bullet \vec{b})^2]/[1 + b^2/\Gamma^2 + (\vec{v} \bullet \vec{b})^2]$$

$$a_1(^-_{fast}) \le a_2(^-_{Alfven}) \le a_3(^-_{slow}) \le a_4(=v_x) \le a_5(^+_{slow}) \le a_6(^+_{Alfven}) \le a_7(^+_{fast})$$

3. Eigen-Structure

Analytic forms of Eigenvectors

$D_i = b_x/(a_i - v_x)$, $Y_{1i} = [a_i(\vec{v} \times \vec{b}|_z) - b_y]$, $Y_{2i} = -[a_i(\vec{v} \times \vec{b}|_y) + b_z]$, $Y_i = \sqrt{Y_{1i}^2 + Y_{2i}^2}$

$AA_i = (a_i - a^{+}_{Alfven})(a_i - a^{-}_{Alfven})(\Gamma^2 + b^2)$: Alfven mode equation

For fast modes, eigenvectors are renormalized with $(a_i - v_x)^2 AA_i$.

All componnents are divided by $(a_i - v_x)^2 AA_i$.

Case1 — $(a_i - v_x)^2$; Case2&3 — AA_i

$G_{1i} = Y_{1i}/AA_i$, $G_{2i} = Y_{2i}/AA_i$,

$G_{i3} = (\vec{v} \times \vec{b}|_x)/AA_i = (G_{1i}v_z - G_{2i}v_y)/(1 - a_i v_x)$, $C_i = 1/(a_i - v_x)^2$.

For slow modes and Alfven modes, eigenvectors are renormalized with $(a_i - v_x)^2 Y_i$.

Case2&3 — Y_i; Case1 — $(a_i - v_x)^2$

All componnents are divided by $(a_i - v_x)^2 Y_i$.

$G_{1i} = Y_{1i}/Y_i$, $G_{2i} = Y_{2i}/Y_i$, $G_{i3} = (\vec{v} \times \vec{b}|_x)/Y_i = (G_{1i}v_z - G_{2i}v_y)/(1 - a_i v_x)$,

$C_i = AA_i/[(a_i - v_x)^2 Y_i]$.

➡ We put the physically meaningful values into Di, G1i, G2i, Ci, instead of 0/0

Di → Case1; G1i, G2i, Ci → Case2&3

3. Eigen-Structure

Analytic forms of Eigenvectors

Compressible mode

$$\vec{R}_i = [\, R_{i1}/(h\Gamma) \,, R_{i2} \,,\; R_{i3} \,,\; R_{i4} \,,\; R_{i5} \,,\; R_{i6}/\sqrt{\rho h} \,,\; R_{i7}/\sqrt{\rho h}\,]^T$$

$$R_{i1} = -G_{1i}(D_i v_y + b_y) - G_{2i}(D_i v_z + b_z) - C_i(1 - a_i v_x)$$

⋮

$$\vec{L}_i = \frac{1}{2n\widetilde{L}_i}[\, L_{i1} h/\Gamma \,,\; L_{i2} \,,\; L_{i3} \,,\; L_{i4} \,,\; L_{i5} \,,\; L_{i6}\sqrt{\rho h} \,,\; L_{i7}\sqrt{\rho h}\,]$$

$$\begin{aligned}\widetilde{L}_i = &-D_i(1 - a_i^2)[D_i/\Gamma^2 - (\vec{v}\cdot\vec{b})][G_{1i}^2 + G_{2i}^2 - (1 - a_i^2)G_{3i}^2] \\ &- C_i(a_i - v_x)^2[(1 - a_i v_x)C_i + G_{i1}(D_i v_y + b_y) + G_{i2}(D_i v_z + b_z)]\end{aligned}$$

⋮

Alfven mode

$$\vec{R}_i = [\, R_{i1}/(h\Gamma) \,,\; R_{i2} \,,\; R_{i3} \,,\; R_{i4} \,,\; R_{i5} \,, R_{i6}/\sqrt{\rho h} \,,\; R_{i7}/\sqrt{\rho h}\,]^T$$

$$R_{i1} = G_{3i}[D_i^2/\Gamma^2 + 2D_i(\vec{v}\cdot\vec{b}) - b^2]$$

⋮

$$\vec{L}_i = \frac{1}{2\widetilde{L}_i}[\, L_{i1} \,,\; L_{i2} \,,\; L_{i3} \,,\; L_{i4} \,,\; L_{i5} \,,\; L_{i6}\sqrt{\rho h} \,,\; L_{i7}\sqrt{\rho h}\,]$$

$$\widetilde{L}_i = D_i[D_i/\Gamma^2 - (\vec{v}\cdot\vec{b})][1 - G_{3i}^2(1 - a_i^2)]$$

⋮

Entropy mode

$$\vec{R}_i = [\, \tfrac{1}{h\Gamma(1-nc_s^2)} \,,\; v_x \,,\; v_y \,,\; v_z \,,\; 1 \,, 0 \,,\; 0\,]^T$$

$$\vec{L}_i = \frac{\Gamma^2(1 - nc_s^2)}{nc_s^2}[\, h/\Gamma \,,\; v_x \,,\; v_y \,,\; v_z \,,\; -1 \,,\; (b_z/\Gamma^2 + v_z(\vec{v}\cdot\vec{b}))\sqrt{\rho h} \,,\; (b_y/\Gamma^2 + v_y(\vec{v}\cdot\vec{b}))\sqrt{\rho h}\,]$$

4. Equation of State

EOS for fixed adiabatic index

h : specific enthalpy γ : polytrophic index

$\gamma = 5/3$ for sub-relativistic

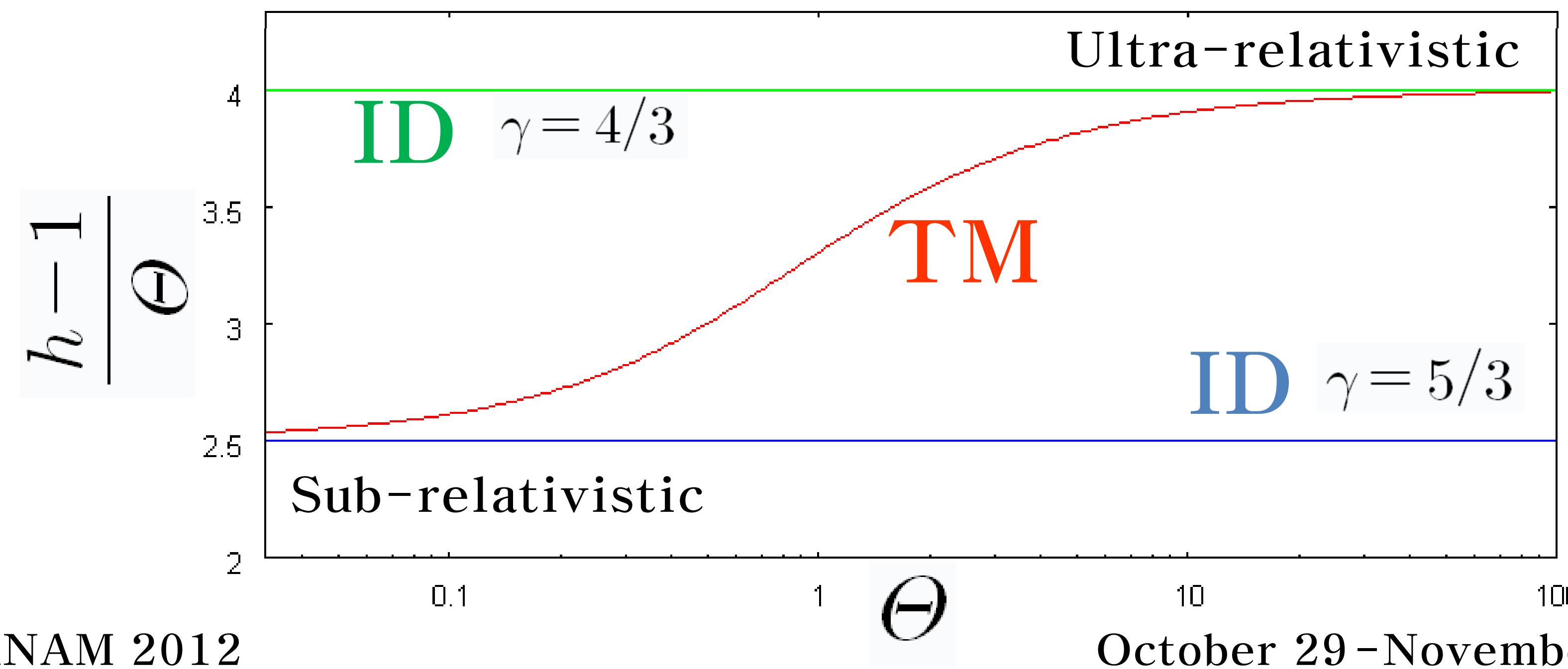
$\gamma = 4/3$ for ultra-relativistic

$$h = 1 + \frac{\gamma\Theta}{\gamma - 1}$$ ID

$$\Theta = p/\rho$$

EOS which models for relativistic perfect gas suggested by Taub(1948)

$$h = \frac{5}{2}\Theta + \frac{3}{2}\sqrt{\Theta^2 + \frac{9}{4}}$$ TM



5. Numerical Tests

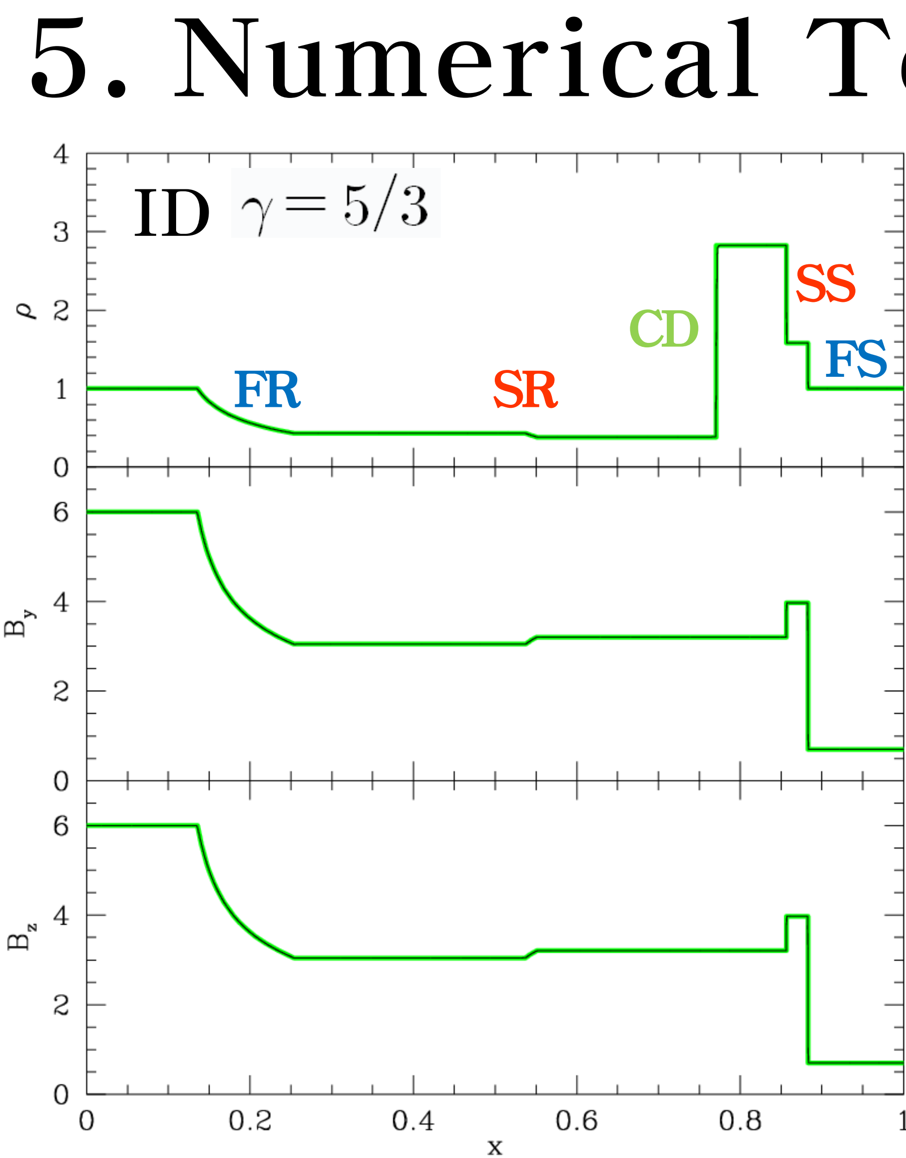
Using the analytic expressions of eigenvalues and eigenvectors,

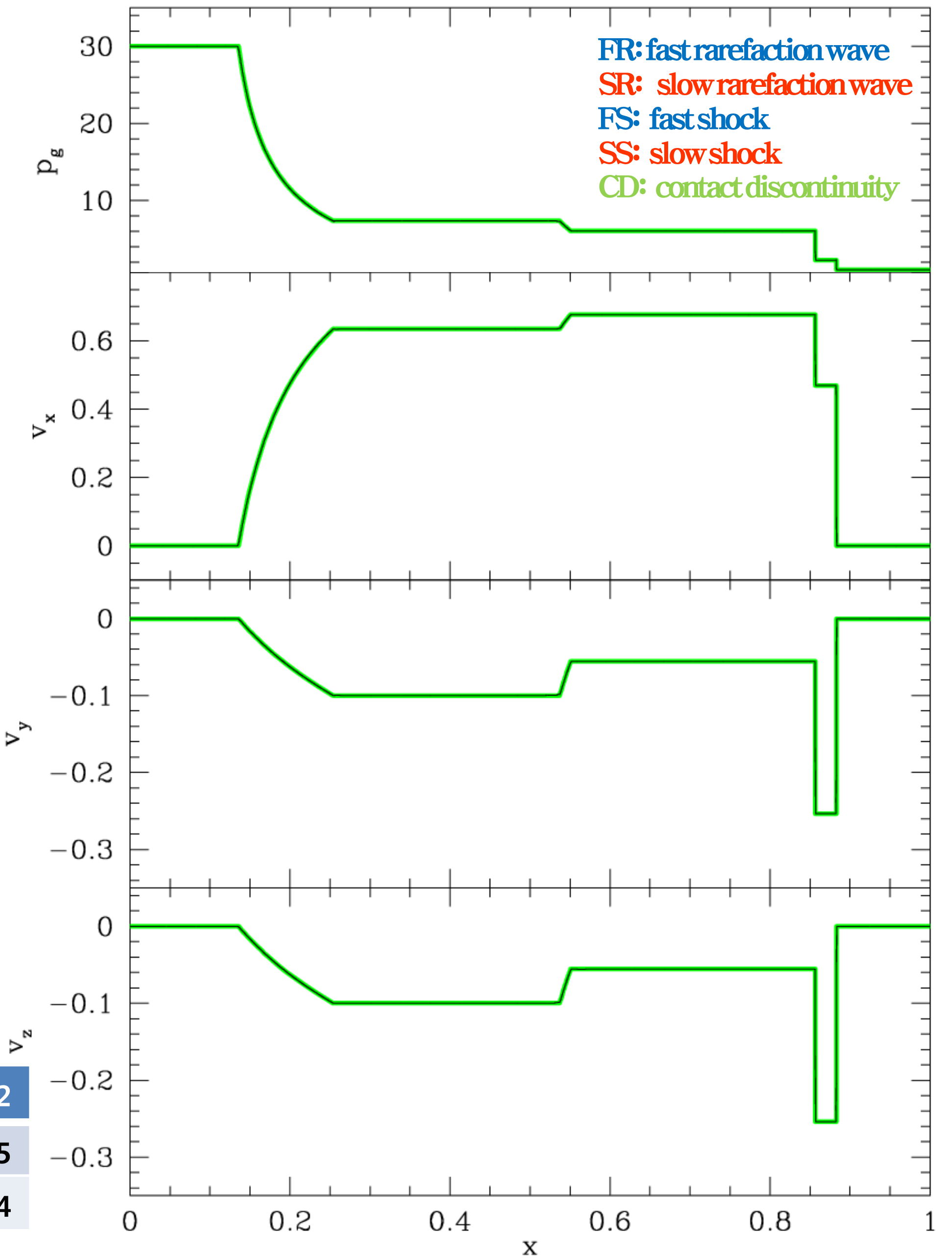
⇒ A RMHD code based on the TVD scheme was built with ID and TM EOS

⇒ One-dimensional shock tube tests

		ρ	vx	vy	vz	By	Bz	p	nstep=8192
test1	L	1	0	0	0	6	6	30	Bx=5
	R	1	0	0	0	0.7	0.7	1	tend=0.4
test2	L	1	0	0.3	0.4	6	2	5	Bx=1
	R	0.9	0	0	0	5	2	5.3	tend=0.375

5. Numerical Tests

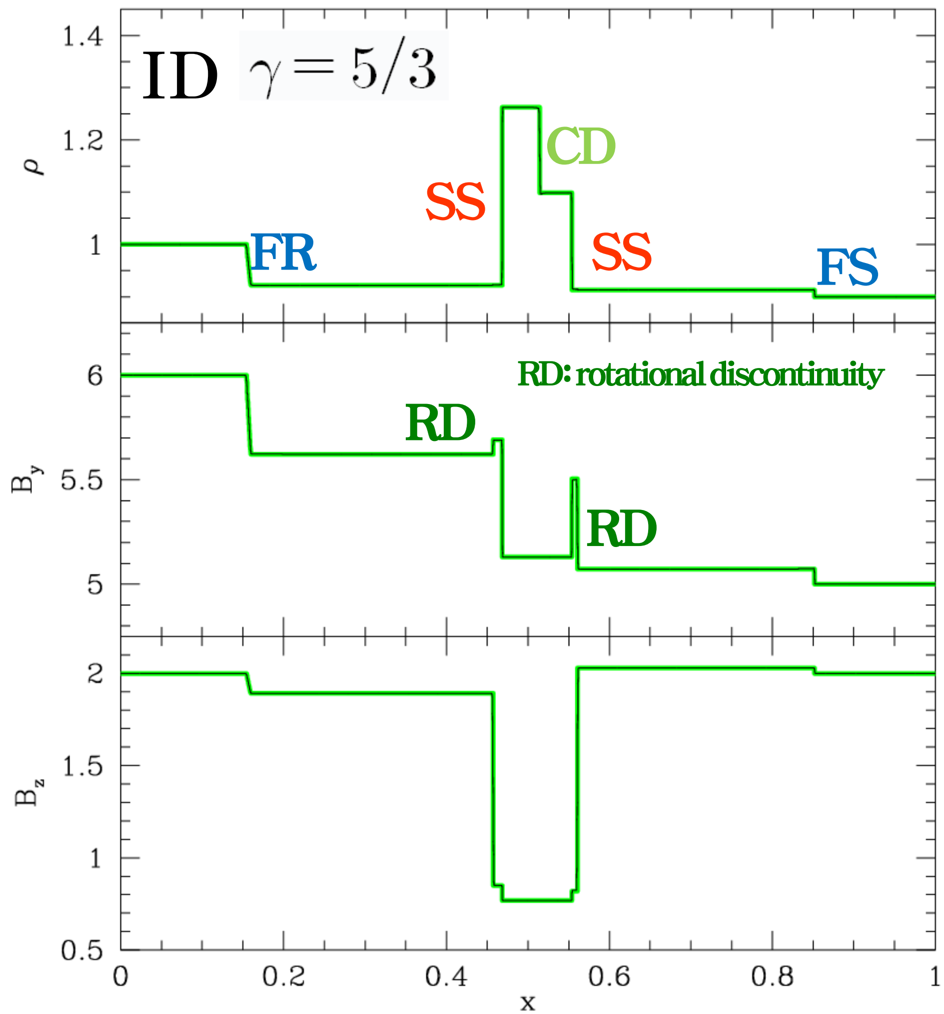


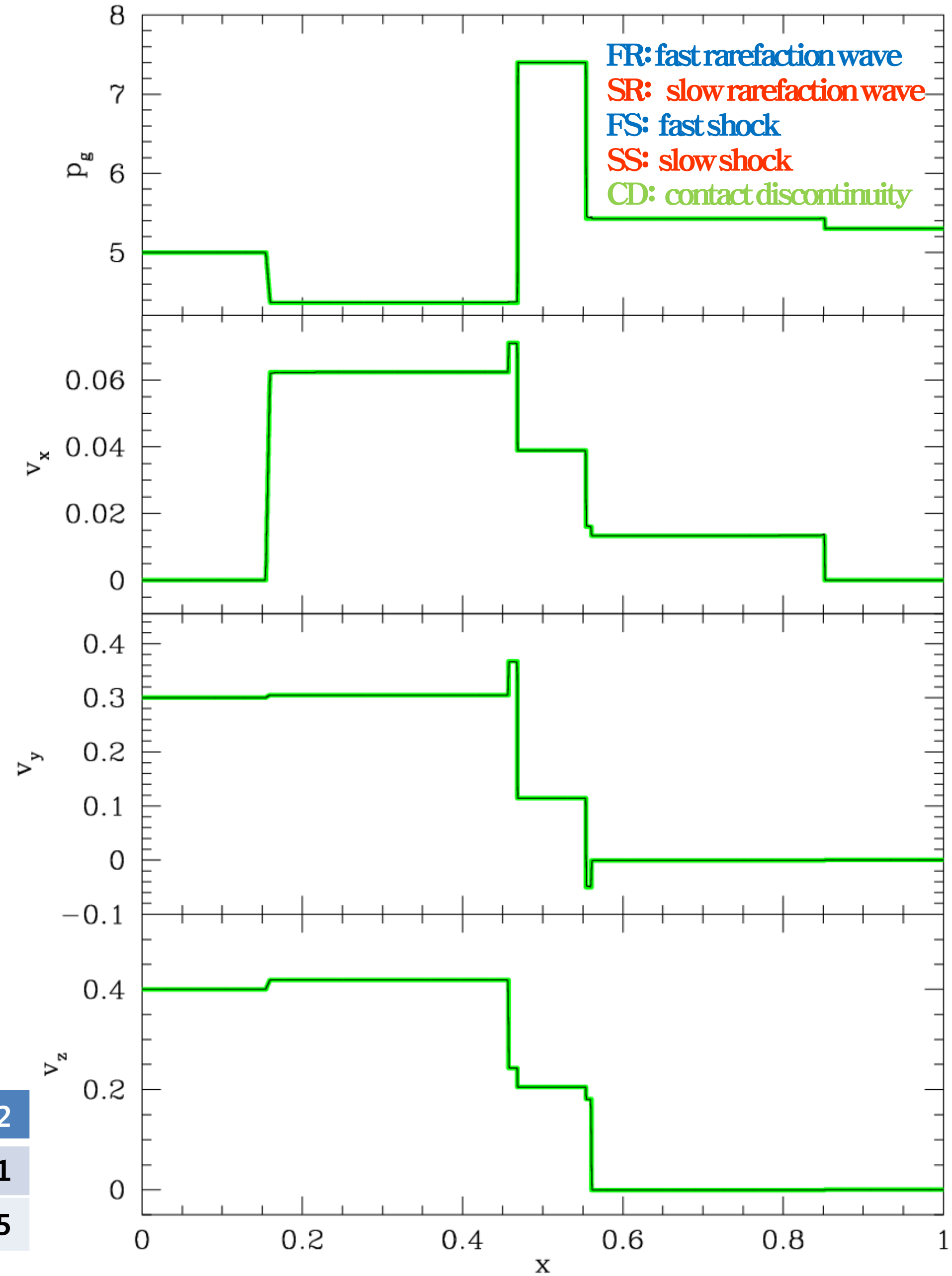


		ρ	vx	vy	vz	By	Bz	p	nstep=8192
test1	L	1	0	0	0	6	6	30	Bx=5
	R	1	0	0	0	0.7	0.7	1	tend=0.4

5. Numerical Tests

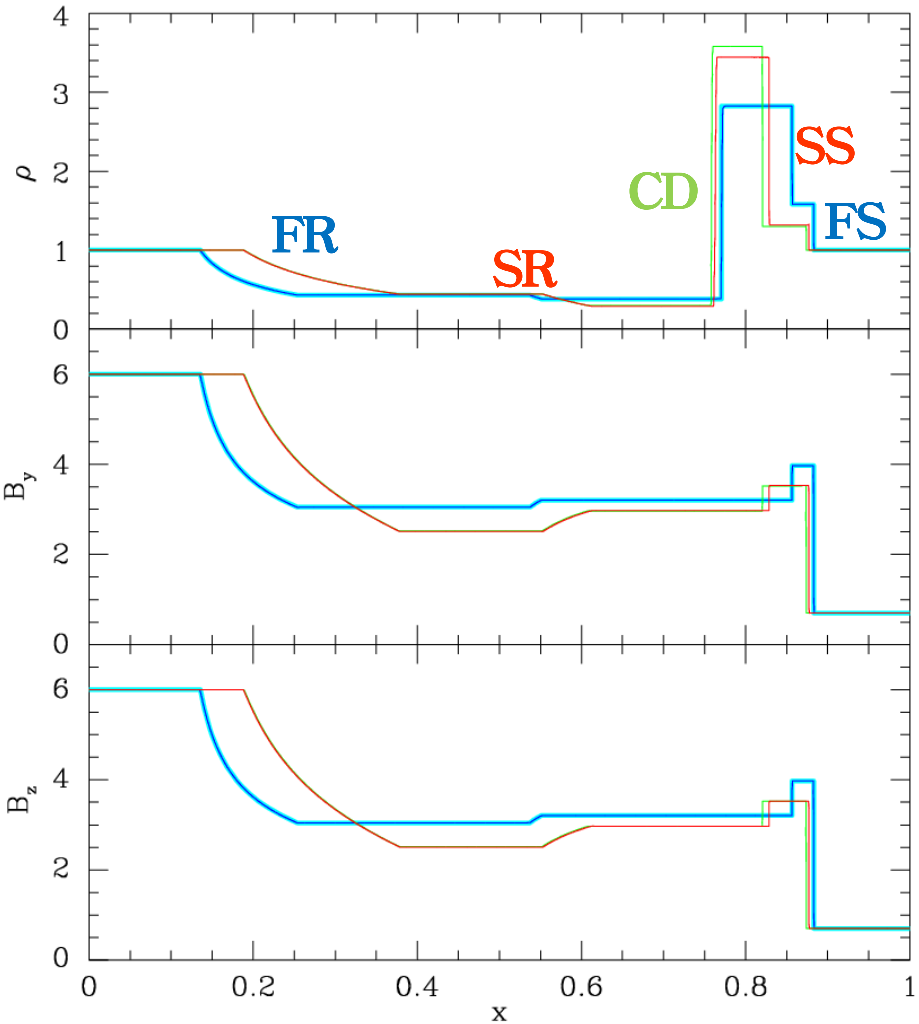
Exact sol. (Bruno 2006)
TVD scheme

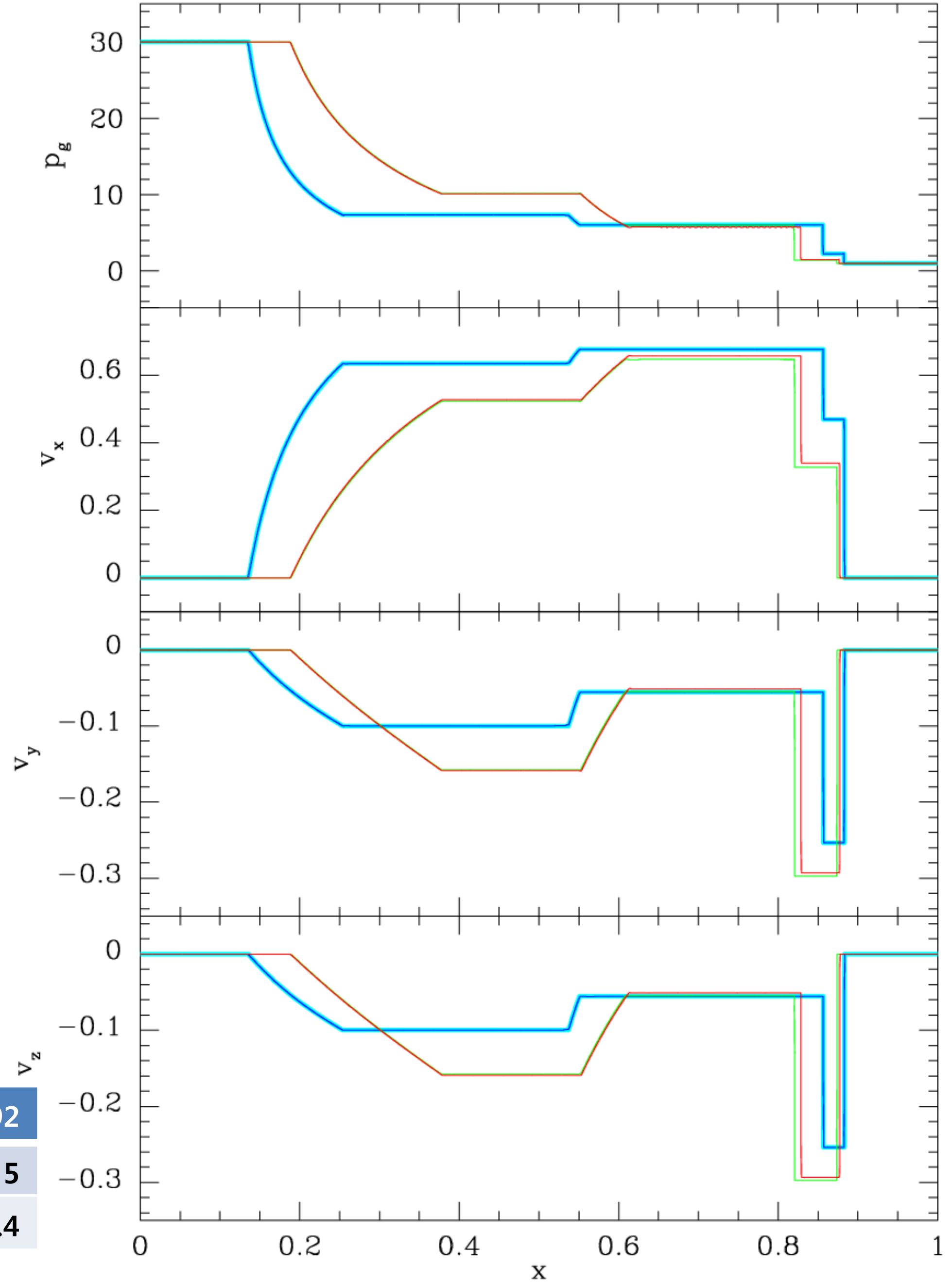




		ρ	vx	vy	vz	By	Bz	p	nstep=8192
test2	L	1	0	0.3	0.4	6	2	5	Bx=1
	R	0.9	0	0	0	5	2	5.3	tend=0.375

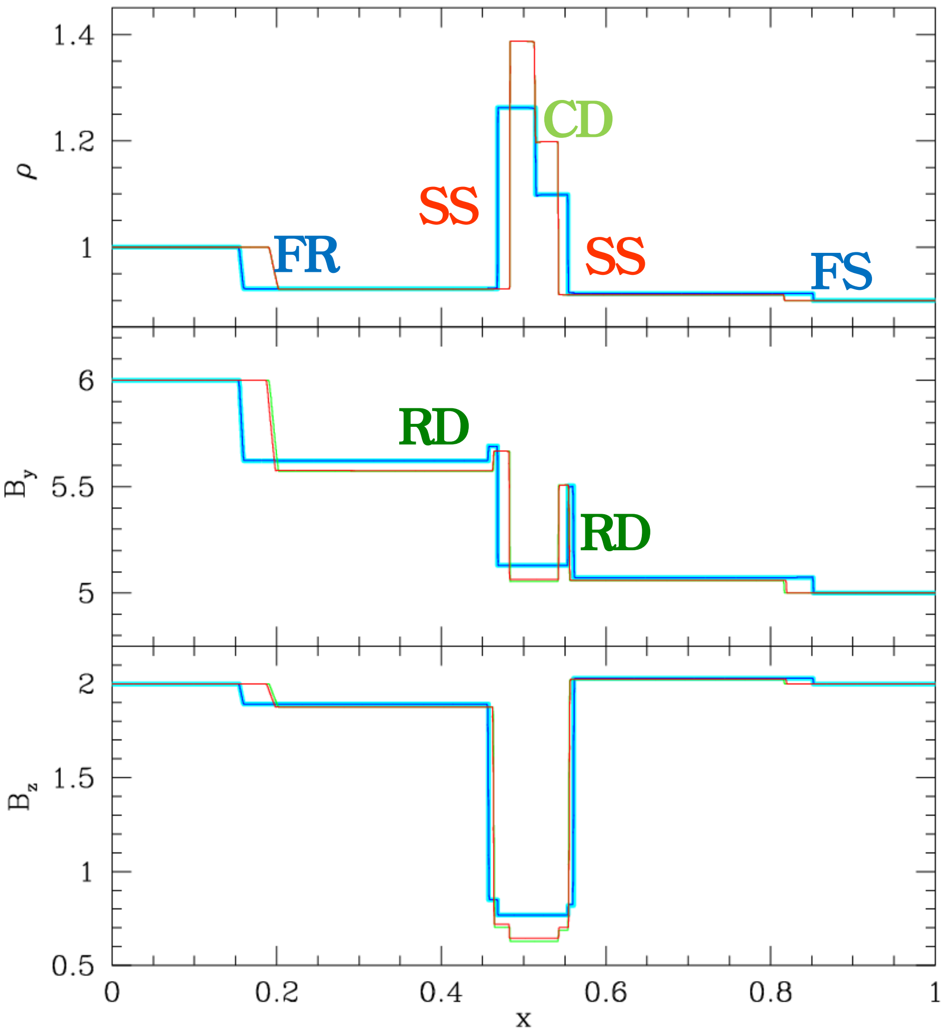
5. Numerical Tests

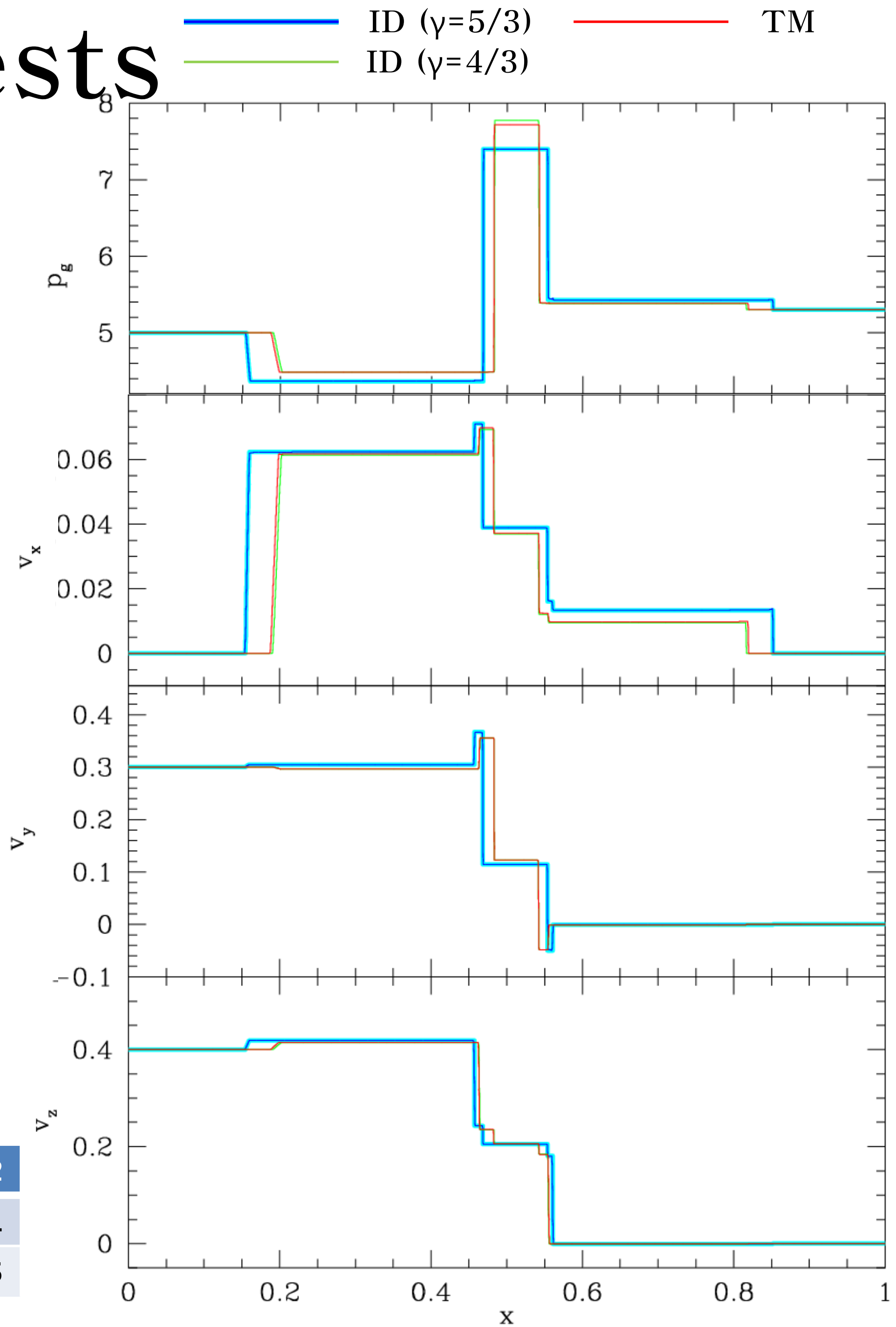




		ρ	vx	vy	vz	By	Bz	p	nstep=8192
test1	L	1	0	0	0	6	6	30	Bx=5
	R	1	0	0	0	0.7	0.7	1	tend=0.4

5. Numerical Tests





		ρ	vx	vy	vz	By	Bz	p	nstep=8192
test2	L	1	0	0.3	0.4	6	2	5	Bx=1
	R	0.9	0	0	0	5	2	5.3	tend=0.375

6. Summary

For building RMHD codes based on upwind schemes

Analytic forms for the eigenvalues and eigenvectors

Degeneracy issue

EOS for fixed adiabatic index & EOS for relativistic perfect gas

Successfully built an RMHD code based on an upwind scheme