Relativistic Hydrodynamic Code for Multi-component Fluids

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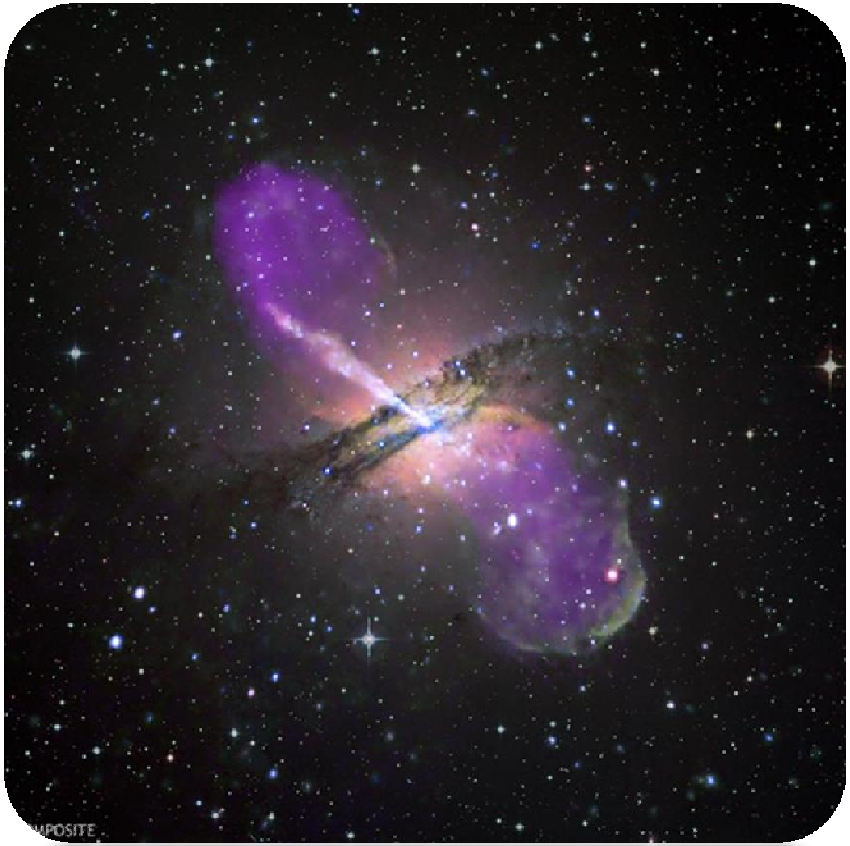
CONTENTS

1. Introduction

1) What are the Relativistic fluids



Centaurus A (NGC 5128)

Jets from Galatic Center

Gamma-ray Bursts

Black Hole accretion Disks

1. Introduction

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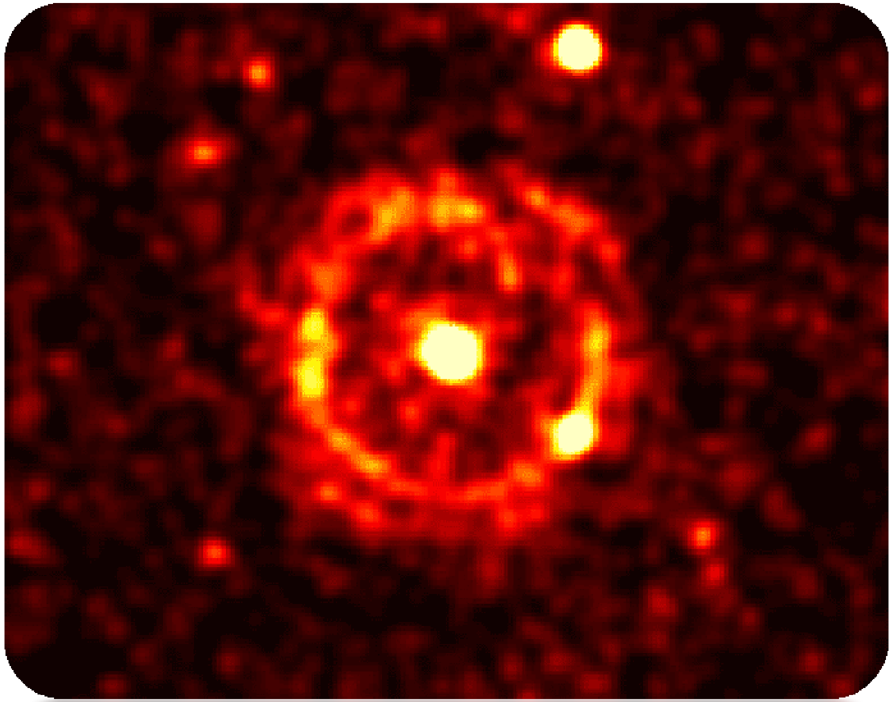

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1. Introduction

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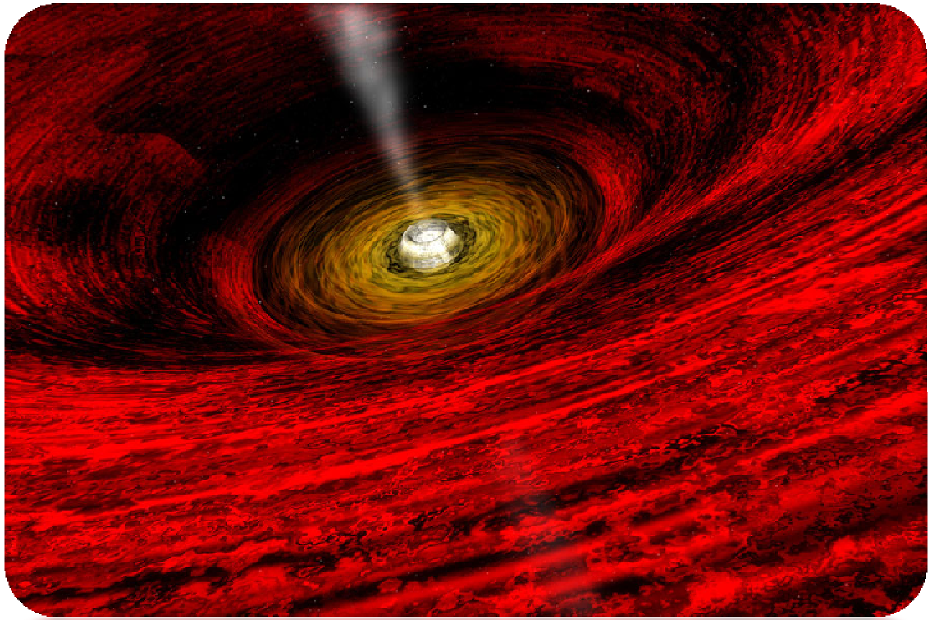

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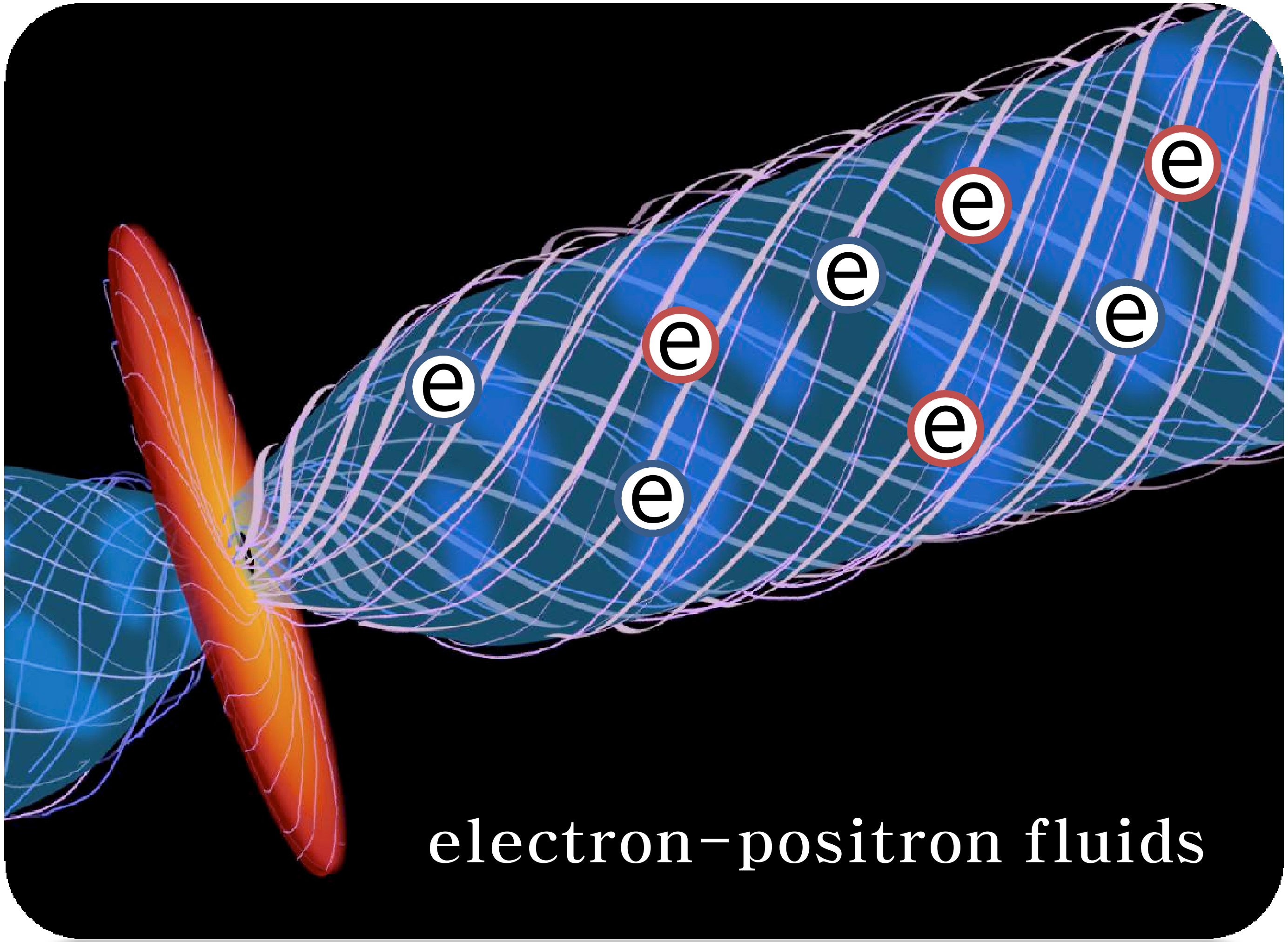
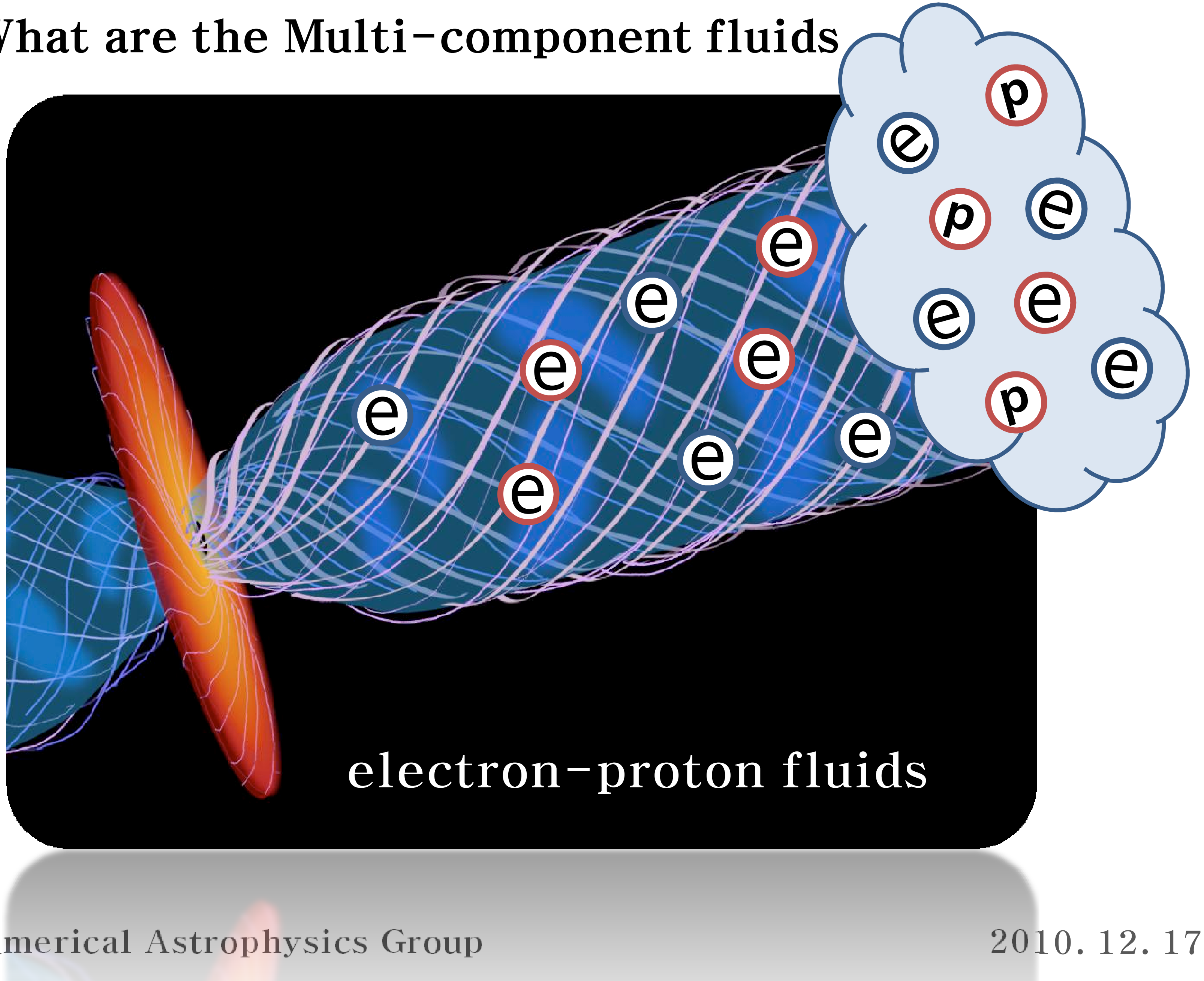
1. Introduction

2) What are the Multi-component fluids



1. Introduction

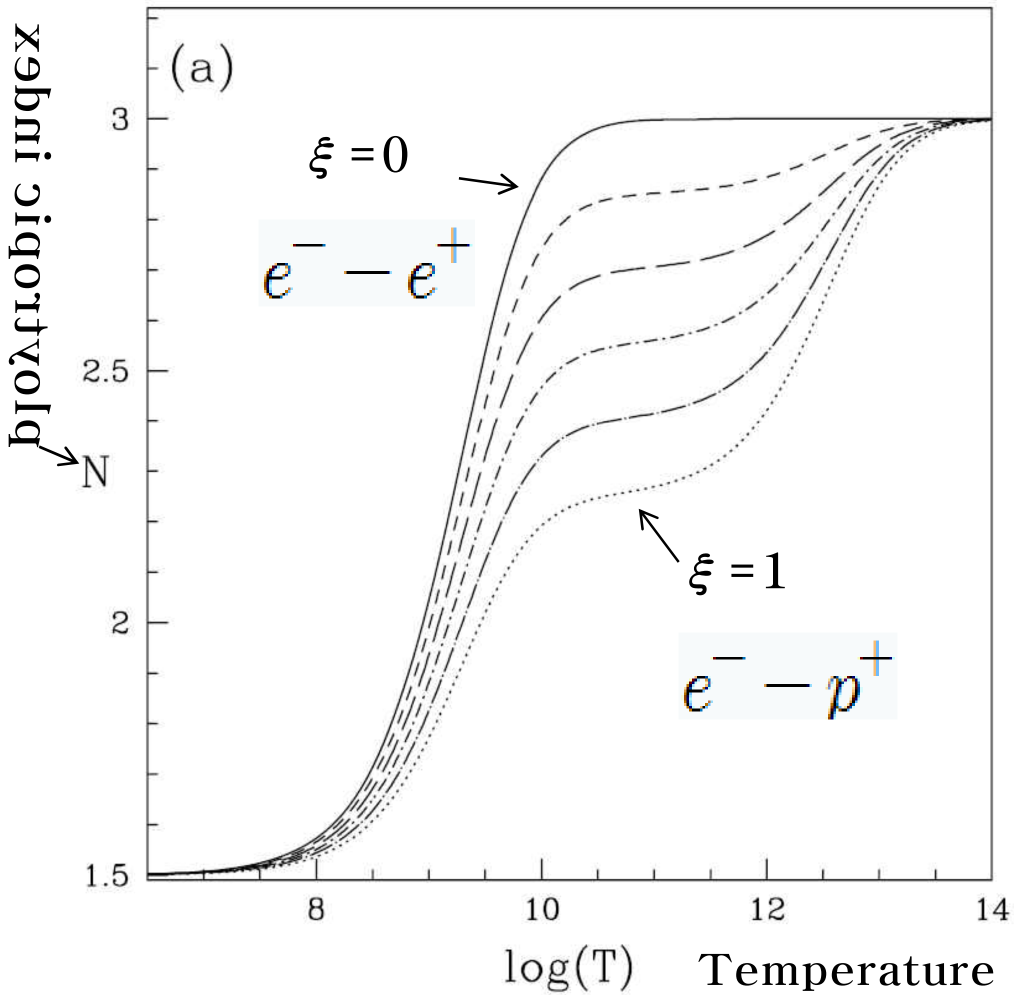
2) What are the Multi-component fluids



1. INTRODUCTION

2) What are the Multi-component fluids

Chattopadhyay and Ryu (2009)



Temperature

$$\xi = \frac{n_{p^+}}{n_{e^-}}$$

relative proportion of proton

ξ = 0, 0.2, 0.4, 0.6, 0.8, 1

$e^- - e^+$ *fluid* $e^- - p^+$ *fluid*

2. Basic Equations

1) Equation of state

$$h = \frac{K_3(\frac{1}{\Theta})}{K_2(\frac{1}{\Theta})}$$

h: specific enthalpy

The correct EOS for single-component fluids in relativistic regime

K_2, K_3 : modified Bessel function of the second kind of order two and three
(Synge 1957)

$$\Theta = \frac{p}{\rho}$$

Θ : temperature-like variable
p : pressure
ρ : total mass density

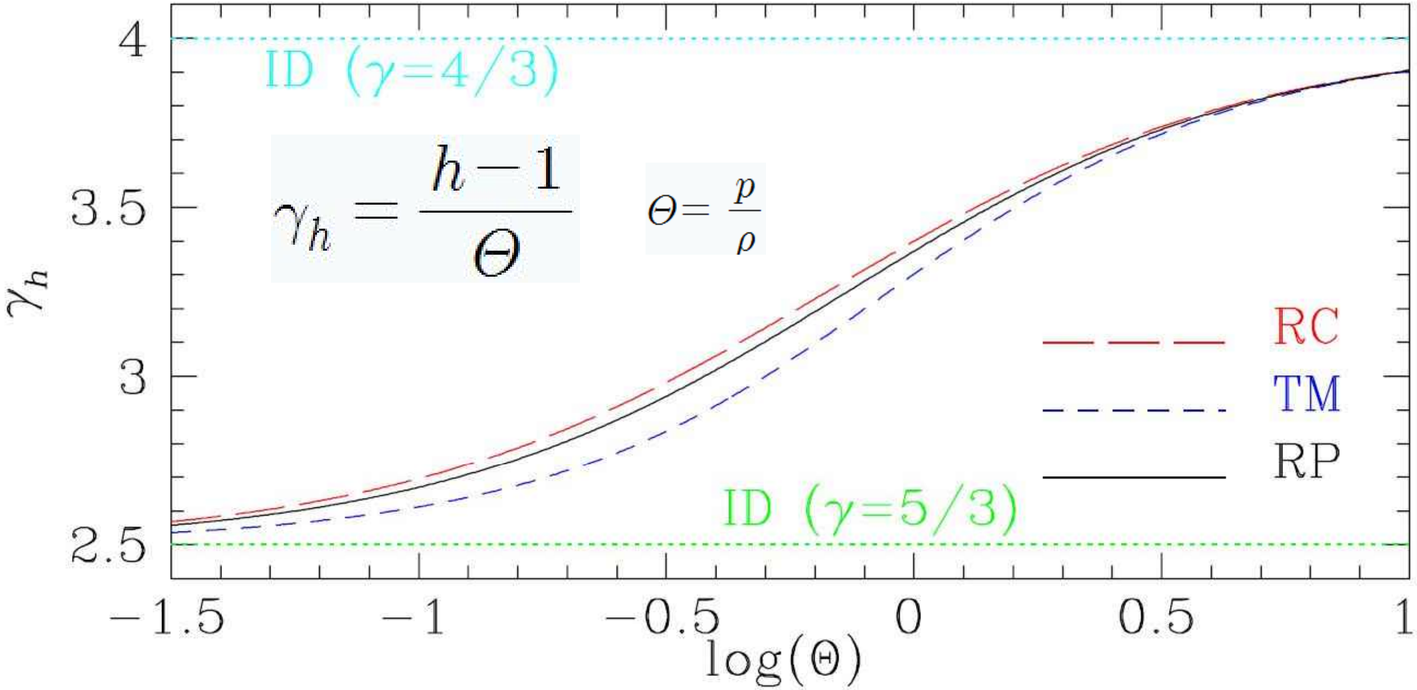
$$h = 2\frac{6\Theta^2 + 4\Theta + 1}{3\Theta + 2}$$

The approximate EOS for single-component fluids
(Ryu et al 2006)

2. Basic Equations

1) Equation of state

Ryu et al. (2006)



RC : the approximate EOS (Ryu et al 2006)
RP : the correct EOS (Synge 1957)
TM : another approximate EOS (Mathews 1971)

$$\frac{|h_{\rm RC} - h_{\rm RP}|}{h_{\rm RP}} \lesssim 0.8\%$$

h: specific enthalpy

2. BASIC EQUATIONS

1) Equation of state

Chattopadhyay and Ryu (2009) $n = \Sigma n_i = n_{e^-} + n_{e^+} + n_{p^+},$

$$n_{e^-} = n_{e^+} + n_{p^+} \quad \Rightarrow \quad n = 2n_{e^-} \quad \text{and} \quad n_{e^+} = n_{e^-}(1-\xi)$$

$$e = \Sigma e_i = \Sigma\left[n_i m_i c^2 + p_i\left(\frac{9p_i + 3n_i m_i c^2}{3p_i + 2n_i m_i c^2}\right)\right] \quad e = n_{e^-} m_e c^2 f$$

$$f = (2-\xi)\left[1 + \Theta'\left(\frac{9\Theta' + 3}{3\Theta' + 2}\right)\right] + \xi\left[\frac{1}{\eta} + \Theta'\left(\frac{9\Theta' + 3/\eta}{3\Theta' + 2/\eta}\right)\right] \quad \eta = m_e/m_p$$

$$p = \Sigma p_i = 2n_{e^-} kT$$

$$\rho = \Sigma n_i m_i = n_{e^-} m_e\left\{2 - \xi\left(1 - \frac{1}{\eta}\right)\right\} \qquad \xi = \frac{n_{p^+}}{n_{e^-}}$$

$$h = \frac{e+p}{\rho}$$

$$\Theta' = \frac{kT}{m_e c^2} = \frac{p}{\rho - \rho_p(1-\eta)} \qquad \eta = \frac{m_e}{m_p}$$

e : mass-energy density
p : pressure
ρ : total mass density
h : specific enthalpy
Θ : temperature-like variable

2. Basic Equations

1) Equation of state

$c \equiv 1$ speed of light

$$h = 1 + \frac{\Theta}{k}\frac{k-\eta}{1-\eta}\frac{9\Theta+3k}{3\Theta+2k} + \eta\frac{\Theta}{k}\frac{1-k}{1-\eta}\frac{9\eta\Theta+3k}{3\eta\Theta+2k} + \Theta$$

$$k = 1 - \zeta(1-\eta) \qquad \eta = \frac{m_e}{m_p} \approx \frac{1}{2000}$$

$$\zeta = \frac{\rho_p}{\rho} = \frac{\xi}{\xi + \eta(2-\xi)} \qquad \xi = \frac{n_{p^+}}{n_{e^-}}$$

ρ :total mass density

ρ_p :proton mass density

$$\Theta = \frac{p}{\rho} = k\Theta'$$

if $\rho_p = 0$ or $\rho_p = \rho$ it becomes single-component case

$$\Rightarrow \quad h = 2\frac{6\Theta^2 + 4\Theta + 1}{3\Theta + 2}$$

(Ryu et al 2006)

2. Basic Equations

1) Equation of state

Single-component (Ryu et al 2006)

$$n = \rho\frac{\partial h}{\partial p} - 1, \qquad c_s^2 = -\frac{\rho}{nh}\frac{\partial h}{\partial \rho}$$

$$n = 3\frac{9\Theta^2 + 12\Theta + 2}{(3\Theta + 2)^2}, \qquad c_s^2 = \frac{\Theta(3\Theta + 2)(18\Theta^2 + 24\Theta + 5)}{3(6\Theta^2 + 4\Theta + 1)(9\Theta^2 + 12\Theta + 2)}$$

n: ploytropic index c_s: sound speed

Multi-component

$$n = \frac{3}{k}\frac{k-\eta}{1-\eta}\frac{9\Theta^2 + 12k\Theta + 2k^2}{(3\Theta + 2k)^2} + \eta\frac{3}{k}\frac{1-k}{1-\eta}\frac{3}{k}\frac{9\eta^2\Theta^2 + 12\eta k\Theta + 2k^2}{(3\eta\Theta + 2k)^2}$$

$$c_t^2 = -\frac{\rho}{nh}\frac{\partial h}{\partial \rho} = \frac{1}{nh}[(1+n)\Theta - (1-k)\frac{\partial h}{\partial k}]$$

c_t: total sound speed

$$c_p^2 = -\frac{\rho_p}{nh}\frac{\partial h}{\partial \rho_p} = \frac{(1-k)}{nh}\frac{\partial h}{\partial k}$$

c_p: proton sound speed

$$\frac{\partial h}{\partial k} = \frac{3\Theta}{k^2(1-\eta)}\frac{9\eta\Theta^2 + 12\eta k\Theta - 3k^2\Theta + 2\eta k^2}{(3\Theta + 2k)^2} - \frac{3\eta\Theta}{k^2(1-\eta)}\frac{9\eta^2\Theta^2 + 12\eta k\Theta - 3\eta k^2\Theta + 2k^2}{(3\eta\Theta + 2k)^2}$$

$$c_s^2 = c_t^2 + c_p^2 = \frac{(1+n)}{nh}\Theta$$

2. Basic Equations

2) Conservation equations

Single-component

(Ryu et al 2006)

$$\frac{\partial D}{\partial t}+\frac{\partial}{\partial x_j}(Dv_j)=0$$

$$\frac{\partial M_i}{\partial t}+\frac{\partial}{\partial x_j}(M_i v_j+p\delta_{ij})=0$$

$$\frac{\partial E}{\partial t}+\frac{\partial}{\partial x_j}[(E+p)v_j]=0$$

$D=\Gamma\rho$ mass density

$M_i=\Gamma^2\rho h v_i$ momentum density

$E=\Gamma^2\rho h-p$ total energy density

$\Gamma=\frac{1}{\sqrt{1-v^2}}$ Lorentz factor

$v^2=v_x^2+v_y^2+v_z^2$

Multi-component

$$\frac{\partial D}{\partial t}+\frac{\partial}{\partial x_j}(Dv_j)=0$$

$$\frac{\partial D_p}{\partial t}+\frac{\partial}{\partial x_j}(D_p v_j)=0$$

$$\frac{\partial M_i}{\partial t}+\frac{\partial}{\partial x_j}(M_i v_j+p\delta_{ij})=0$$

$$\frac{\partial E}{\partial t}+\frac{\partial}{\partial x_j}[(E+p)v_j]=0$$

$D=\Gamma\rho$ total mass density

$D_p=\Gamma\rho_p$ proton mass density

p : pressure
ρ : total mass density
ρ_p : proton mass density
v : fluid velocity

2. Basic Equations

2) Conservation equations

$$\frac{\partial \vec{q}}{\partial t} + \frac{\partial \vec{F}}{\partial x_j} = 0 \qquad \vec{q} = \begin{pmatrix} D \\ D_p \\ M_i \\ E \end{pmatrix} \qquad \vec{F_j} = \begin{pmatrix} Dv_j \\ D_p v_j \\ M_i v_j + p\delta_{ij} \\ (E+p)v_j \end{pmatrix}$$

$\vec{q}$: state vector

$\vec{F}$: flux vector

$$\frac{\partial \vec{q}}{\partial t} + A_j \frac{\partial \vec{q}}{\partial x_j} = 0 \qquad A_j = \frac{\partial \vec{F_j}}{\partial \vec{q}} = \frac{\partial \vec{F_j}}{\partial \vec{u}} \frac{\partial \vec{u}}{\partial \vec{q}} \qquad \vec{u} = \begin{pmatrix} \rho \\ \rho_p \\ v_i \\ p \end{pmatrix}$$

A_j : the 6x6 Jacobian matrix

$\vec{u}$: parameter vector

We use the TVD scheme to solve the conservation equations

The code based on the TVD scheme utilizes the Eigen-structure of the Jacobian matrix

3. Eigen-Structure

1) One velocity component

Eigen values

$$a_1 = \frac{v_x - c_s}{1 - c_s v_x}, \qquad a_2 = a_3 = v_x, \qquad a_4 = \frac{v_x + c_s}{1 + c_s v_x}$$

⇒ Same as single-component case!

Single-component ⇒ **Multi-component**

(Ryu et al 2006)

$$c_s^2 = -\frac{\rho}{nh}\frac{\partial h}{\partial \rho}$$

$$c_t^2 = -\frac{\rho}{nh}\frac{\partial h}{\partial \rho}$$ total sound speed

$$c_p^2 = -\frac{\rho_p}{nh}\frac{\partial h}{\partial \rho_p}$$ proton sound speed

$$c_s^2 = c_t^2 + c_p^2$$

3. Eigen-Structure

1) One velocity component

Right Eigen vectors

$$\overrightarrow{R_1}=\begin{bmatrix}1\\ \frac{\rho_p}{\rho}\\ \Gamma h(v_x-c_s)\\ \Gamma h(1-c_s v_x)\end{bmatrix}\quad \overrightarrow{R_2}=\begin{bmatrix}0\\ 1\\ \Gamma\rho h n\frac{c_p^2}{\rho_p}v_x\\ \Gamma\rho h n\frac{c_p^2}{\rho_p}\end{bmatrix}\quad \overrightarrow{R_3}=\begin{bmatrix}1\\ 0\\ \Gamma h v_x(1-nc_t^2)\\ \Gamma h(1-nc_t^2)\end{bmatrix}\quad \overrightarrow{R_4}=\begin{bmatrix}1\\ \frac{\rho_p}{\rho}\\ \Gamma h(v_x+c_s)\\ \Gamma h(1+c_s v_x)\end{bmatrix}$$

Left Eigen vectors

$$\overrightarrow{L_1}=-\frac{1}{2hnc_s^2}\left[h(1-nc_t^2),\ -\rho h n\frac{c_p^2}{\rho_p},\ \Gamma(v_x+nc_s),\ -\Gamma(1+nc_s v_x)\right]$$

$$\overrightarrow{L_2}=\frac{1}{\rho h n c_s^2}\left[\rho_p h(1-nc_t^2),\ \rho h n c_t^2,\ \Gamma\rho_p v_x,\ -\Gamma\rho_p\right]$$

$$\overrightarrow{L_3}=\frac{1}{hnc_s^2}\left[h(1+nc_p^2),\ -\rho h n\frac{c_p^2}{\rho_p},\ \Gamma v_x,\ -\Gamma\right]$$

$$\overrightarrow{L_4}=-\frac{1}{2hnc_s^2}\left[h(1-nc_t^2),\ -\rho h n\frac{c_p^2}{\rho_p},\ \Gamma(v_x-nc_s),\ -\Gamma(1-nc_s v_x)\right]$$

3. EIGEN-STRUCTURE

2) Three velocity components

Eigen values

$$a_1 = \frac{(1-c_s^2)v_x - c_s/\Gamma\sqrt{Q}}{1-c_s^2 v^2} \qquad a_2 = a_3 = a_4 = a_5 = v_x \qquad a_6 = \frac{(1-c_s^2)v_x + c_s/\Gamma\sqrt{Q}}{1-c_s^2 v^2} \qquad Q = 1 - v_x^2 - c_s^2(v_y^2 + v_z^2)$$

Right Eigen vectors

$$\overrightarrow{R_1} = \begin{bmatrix} \frac{1-a_1 v_x}{\Gamma} \\ \frac{\rho_p}{\rho}\frac{1-a_1 v_x}{\Gamma} \\ a_1 h(1-v_x^2) \\ h(1-a_1 v_x)v_y \\ h(1-a_1 v_x)v_z \\ h(1-v_x^2) \end{bmatrix} \quad \overrightarrow{R_2} = \frac{\Gamma^2}{nc_s^2(1-v_x^2)} \begin{bmatrix} \frac{n(v_y^2+v_z^2)}{\Gamma h}\frac{c_p^2}{\rho_p} \\ \frac{nc_p^2(v_y^2+v_z^2)+(1-v_x^2)}{\Gamma\rho h} \\ nc_p^2/\rho_p[2(v_y^2+v_z^2)-(1-v_x^2)]v_x \\ nc_p^2/\rho_p(v_y^2+v_z^2)v_y \\ nc_p^2/\rho_p(v_y^2+v_z^2)v_z \\ nc_p^2/\rho_p[2(v_y^2+v_z^2)-(1-v_x^2)] \end{bmatrix} \quad \overrightarrow{R_3} = \frac{\Gamma^2}{nc_s^2(1-v_x^2)} \begin{bmatrix} \frac{nc_t^2(v_y^2+v_z^2)+(1-v_x^2)}{\Gamma h} \\ \frac{\rho_p}{\rho}\frac{nc_t^2(v_y^2+v_z^2)}{\Gamma h} \\ [2nc_t^2(v_y^2+v_z^2)+(1-nc_t^2)(1-v_x^2)]v_x \\ [nc_t^2(v_y^2+v_z^2)+(1-v_x^2)]v_y \\ nc_t^2(v_y^2+v_z^2)+(1-v_x^2)]v_z \\ 2nc_t^2(v_y^2+v_z^2)+(1-nc_t^2)(1-v_x^2) \end{bmatrix}$$

$$\overrightarrow{R_4} = \frac{1}{(1-v_x^2)} \begin{bmatrix} \frac{v_y}{\Gamma h} \\ \frac{\rho_p}{\rho}\frac{v_y}{\Gamma h} \\ 2v_x v_y \\ 1-v_x^2+v_y^2 \\ v_y v_z \\ 2v_y \end{bmatrix} \quad \overrightarrow{R_5} = \frac{1}{(1-v_x^2)} \begin{bmatrix} \frac{v_z}{\Gamma h} \\ \frac{\rho_p}{\rho}\frac{v_z}{\Gamma h} \\ 2v_x v_z \\ v_y v_z \\ 1-v_x^2+v_z^2 \\ 2v_z \end{bmatrix} \quad \overrightarrow{R_6} = \begin{bmatrix} \frac{1-a_6 v_x}{\Gamma} \\ \frac{\rho_p}{\rho}\frac{1-a_6 v_x}{\Gamma} \\ a_6 h(1-v_x^2) \\ h(1-a_6 v_x)v_y \\ h(1-a_6 v_x)v_z \\ h(1-v_x^2) \end{bmatrix}$$

3. Eigen-Structure

2) Three velocity components

Left Eigen vectors

$$\overrightarrow{L_1} = \frac{1}{2hnQ(a_1 - v_x)^2}\begin{bmatrix} -\frac{h}{\Gamma}(1-a_1 v_x)(1-nc_t^2) \\ \frac{\rho h n}{\Gamma}\frac{c_p^2}{\rho_p}(1-a_1 v_x) \\ na_1(1-c_s^2 v^2) + a_1(1+nc_s^2)v_x^2 - (1+n)v_x \\ -(1+nc_s^2)(1-a_1 v_x)v_y \\ -(1+nc_s^2)(1-a_1 v_x)v_z \\ (1+nc_s^2 v^2) + (1-c_s^2)nv_x^2 - a_1(1+n)v_x \end{bmatrix}^T$$

$$Q = 1 - v_x^2 - c_s^2(v_y^2 + v_z^2)$$

$$\overrightarrow{L_2} = \left[\frac{\rho_p h}{\Gamma}(1-nc_t^2),\ \frac{\rho h n c_t^2}{\Gamma},\ \rho_p v_x,\ \rho_p v_y,\ \rho_p v_z,\ -\rho_p\right]$$

$$\overrightarrow{L_4} = [-\Gamma h v_y,\ 0, 0,\ 1,\ 0,\ 0]$$

$$\overrightarrow{L_3} = \left[\frac{h}{\Gamma}(1+nc_p^2),\ -\frac{\rho h n}{\Gamma}\frac{c_p^2}{\rho_p},\ v_x,\ v_y,\ v_z,\ -1\right]$$

$$\overrightarrow{L_5} = [-\Gamma h v_z,\ 0, 0,\ 0,\ 1,\ 0]$$

$$\overrightarrow{L_6} = \frac{1}{2hnQ(a_6 - v_x)^2}\begin{bmatrix} -\frac{h}{\Gamma}(1-a_6 v_x)(1-nc_t^2) \\ \frac{\rho h n}{\Gamma}\frac{c_p^2}{\rho_p}(1-a_6 v_x) \\ na_1(1-c_s^2 v^2) + a_6(1+nc_s^2)v_x^2 - (1+n)v_x \\ -(1+nc_s^2)(1-a_6 v_x)v_y \\ -(1+nc_s^2)(1-a_6 v_x)v_z \\ (1+nc_s^2 v^2) + (1-c_s^2)nv_x^2 - a_6(1+n)v_x \end{bmatrix}^T$$

4. Calculation of Primitive Variables

$$D = \Gamma \rho$$
$$D_p = \Gamma \rho_p$$
$$M_i = \Gamma^2 \rho h v_i$$
$$E = \Gamma^2 \rho h - p$$

conserved variables

- D : total mass density
- Dp: proton mass density
- M : momentum density
- E : total energy density

primitive variables

- p : pressure
- ρ : total mass density
- ρ_p: proton mass density
- v : fluid velocity

Combining these equations with EOS

$$2\sqrt{\Gamma^2-1}\,[3\Gamma^2(3\eta\Gamma(6\Gamma^2-1)E-(\eta+1)(4\Gamma^2-1)C)M^2$$
$$+(\Gamma^2-1)[18\eta\Gamma^3E^3-12(\eta+1)\Gamma^2CE^2+\Gamma C(3(\eta+1)D+5C)E-2DC^2]]$$
$$=M[9\eta\Gamma^4(4\Gamma^2-1)M^2+(\Gamma^2-1)[9\eta\Gamma^2(12\Gamma^2-1)E^2$$
$$-6(\eta+1)\Gamma(8\Gamma^2-1)CE+2C(3(\eta+1)\Gamma^2D+(5\Gamma^2-2)C)]]$$

$$C = D - D_p(1-\eta)$$

It reduces into an equation of 12th power in Γ

$$A_1\Gamma^{12} + A_2\Gamma^{11} + A_3\Gamma^{10} + \cdots = 0$$

$$\Gamma = \frac{1}{\sqrt{1-v^2}}$$ Lorentz factor

$$v^2 = v_x^2 + v_y^2 + v_z^2$$

$A_1, A_2, A_3, \cdots$ are function of D, Dp, M, E

4. Calculation of Primitive Variables

$$A_1\Gamma^{12} + A_2\Gamma^{11} + A_3\Gamma^{10} + \cdots = 0$$

When p = 0

$$\Gamma_u = \frac{1}{\sqrt{1 - M^2/E^2}}$$ upper limit of Γ

When $\rho = \rho_p = 0$

$$\Gamma_l = \frac{\sqrt{2E^2 - M^2 + E\sqrt{4E^2 - 3M^2}}}{2\sqrt{E^2 - M^2}}$$ lower limit of Γ

M : momentum density
E : total energy density

The physically meaningful solution should be between Γ_u and Γ_l

4. Calculation of Primitive Variables

$$A_1\Gamma^{12} + A_2\Gamma^{11} + A_3\Gamma^{10} + \cdots = 0$$

⇨ It can be easily calculated by the Newton-Raphson method

12 roots of Γ

4 are complex and 8 are real
among 8 real, 3 are negative and 5 are positive
among 5 real & positive, always 1 or 2 is between Γ_u and Γ_l
always smaller one is the physical solution

Problem!

The Newton-Raphson method could get a wrong solution

4. Calculation of Primitive Variables

If we find a right solution among two candidates,
the primitive variables can be calculated

$$D = \Gamma\rho$$
$$D_p = \Gamma\rho_p$$
$$M_i = \Gamma^2 \rho h v_i$$
$$E = \Gamma^2 \rho h - p$$

conserved variables

D : total mass density
Dp: proton mass density
M : momentum density
E : total energy density

primitive variables

p : pressure
ρ : total mass density
ρ_p : proton mass density
v : fluid velocity

$$v = \frac{\sqrt{\Gamma^2 - 1}}{\Gamma} \qquad v_x = \frac{M_x}{M} v \qquad v_y = \frac{M_y}{M} v \qquad v_z = \frac{M_z}{M} v$$

$$\rho = \frac{D}{\Gamma} \qquad \rho_p = \frac{D_p}{\Gamma} \qquad p = \frac{M}{v} - E$$

→ Another form of p is being calculated for v=0

4. Calculation of Primitive Variables

$$A_1\Gamma^{12} + A_2\Gamma^{11} + A_3\Gamma^{10} + \cdots = 0$$

We have to find a right solution of Γ

One possibility

We already know the numerical values of the conserved variables by TVD scheme

D, Dp, M, E

Using

$$v = \frac{\sqrt{\Gamma^2 - 1}}{\Gamma} \quad \rho = \frac{D}{\Gamma} \quad \rho_p = \frac{D_p}{\Gamma} \quad p = \frac{M}{v} - E$$

we can get v, ρ, ρ_p, p

And using

$$D = \Gamma\rho \quad D_p = \Gamma\rho_p \quad M_i = \Gamma^2\rho h v_i \quad E = \Gamma^2\rho h - p$$

we can get D, Dp, M, E again!

If D, Dp, M, E are different from the initial value, that Γ is the unphysical solution!

5. SUMMARY

1. There are many high energy phenomena in the universe
2. To understand these relativistic fluids ,
   we study RHD numerically
3. The fluid' s components can be changed
4. To study Multi-component Fluids is important

Outline of the RHD code for multi-component fluids

1. Calculate the Eigen-structure
2. Using TVD scheme, we can get D, Dp, M, E
3. Solve the equation of 12th power in Γ
4. Calculate the primitive variable v, ρ, ρ_p, p

Numerical code will be built up soon!

Thank you