Gravitational Instability of Pressure-Confined Rotating Polytropic Disks

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Introduction

- Star formation and structure formation in galaxies, at least in its early phases, is governed by gravitational instability
- Highly turbulent and clumpy high-redshift galaxy (Elmegreen & Elmegreen 2005, Elmegreen et al., 2008)
  - Accretion from intergalatic medium can have high impact velocity
    - ~0.4 times the circular speed if the gas falls in along cosmological filaments (Keres et al. 2005)
    - A possible source of pressure-confinement for gas rich disk
  - Clumpy disks show constant scale height as a function of disk radius

$$H \sim \frac{\sigma^2}{G\Sigma} \sim \mathrm{const} \qquad T \sim c_s^2 \overset{?}{\sim} \sigma^2 \sim \Sigma \quad \Sigma \sim \rho H \sim \rho$$

$$P \sim \rho T \sim \Sigma^2$$

:Polytropic relation may hold

Previous studies on GI of flattened systems

- Axisymmetric stability of rotating disk with vertical stratification (Goldreich & Lynden-Bell, 1965)
  - $Q_c = 0.676$ for isothermal gas disk
- Stability of shock-compressed layers (Elmegreen & Elmegreen, 1978)
  - Compressed gaseous slabs formed by
    - Shock waves from SN explosions, collisions of molecular clouds, expanding fronts of HII
  - **Saturation of the critical wavelength on the order of the layer thickness**
- Stability of a strongly compressed layer (Lubow & Pringle, 1993)
  - In the limit of strong external pressure, unstable mode shows essentially incompressible features
  - **But not clear physical explanation for why they become unstable**
- Numurical simulations of compressed layer (Umekawa et al., 1999)
  - Fragmentation of unstable disk patch into Jeans-stable Bonnor-Ebert spheres?

Purpose of study

- Stability of vertically stratified rotating disk (with polytropic EOS)
- Clarify the effect of pressure confinement on GI
- Obtain stability criterion for rotating pressure-confined disks

Equilibrium Model

System description

- Initial configuration
  - Disk in vertical hydrostatic equilibrium with truncation
  - Differential rotation accomplished by radial gravitational field
- Shearing box approximation
  - A local co-rotating Cartesian reference frame with angular velocity Ω_0
  - $|x|, |y| << R_0$ (distance from rotation axis)
  - With local shear rate

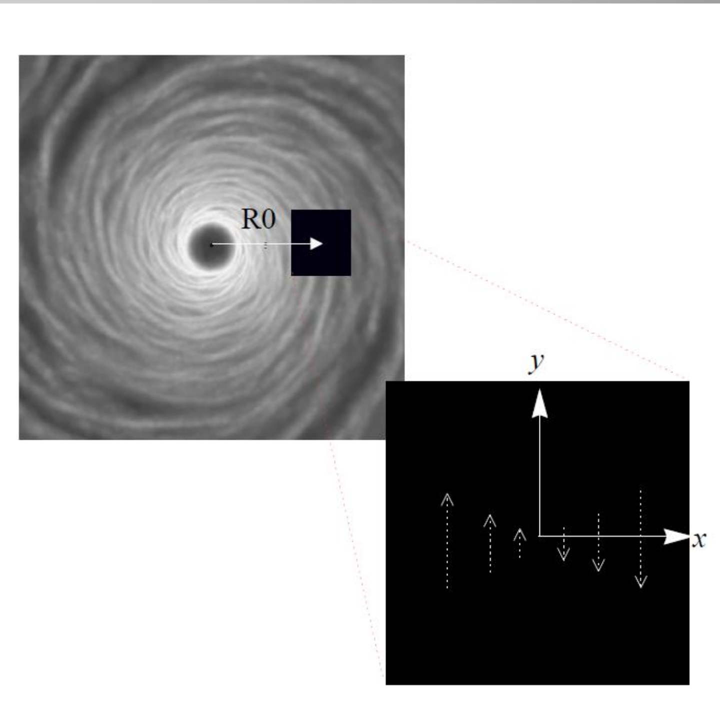
$$q \equiv -\frac{d\ln\Omega}{d\ln R}\Big|_0$$

  - Local epicyclic frequency

$$\kappa_0^2 = (4-2q)\Omega_0^2$$

  - Background flow

$$\mathbf{u}_0 = -q\Omega_0 x\hat{\mathbf{y}}$$



Relevant fluid equations

- Continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0$$

- Momentum equations

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla P - \nabla \psi + 2q\Omega_0^2 x\hat{\mathrm{x}} - 2\mathbf{\Omega_0} \times \mathbf{u}$$

- Poisson's equation

$$\nabla^2 \psi = 4\pi G \rho$$

- Conservation of entropy (adiabatic condition)

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{P}{\rho^\gamma}\right) = 0$$

  - γ : adiabatic index

Vertical structure of a self-gravitating polytropic disk

- Hydrostatic equilibrium

$$\frac{\mathrm{d}\psi_0}{\mathrm{d}z} = -\frac{1}{\rho_0}\frac{\mathrm{d}P_0}{\mathrm{d}z} \qquad \frac{\mathrm{d}^2\psi_0}{\mathrm{d}z^2} = 4\pi G\rho_0$$

- Adopt polytropic relation with polytropic index n (Γ = 1 + 1/n)

$$P_0 = K\rho_0^{1+1/n} = K\rho_0^{\Gamma}$$

- Transform the density and pressure into dimensionless form

$$\rho_0(z) = \rho_{00}\Theta^n(z) \qquad P_0 = K\rho_{00}^{1+1/n}\Theta^{n+1}(z)$$

Vertical structure of a self-gravitating polytropic disk

- Define Dimensionless gravitational acceleration (or surface density as a Lagrangian variable)

$$\mu \equiv \frac{1}{4\pi G \rho_{00} H} \frac{\mathrm{d}\psi_0}{\mathrm{d}z}$$

with scale height defined by

$$H^2 = \frac{K \rho_{00}^{1/n}}{2\pi G \rho_{00}}$$

H: Characteristic height at which density of the disk drops significantly relative to the value at the midplane

- It is straight forward to show that

$$d\psi_0 = -(1+n) K \rho_{00}^{1/n} d\Theta$$

$$\frac{\mathrm{d}\psi_0}{\mathrm{d}z} = -\frac{1}{\rho_0} \frac{\mathrm{d}P_0}{\mathrm{d}z}$$

and from Poisson's equation

$$d\left(\frac{\mathrm{d}\psi_0}{\mathrm{d}z}\right) = 4\pi G \rho_{00} \Theta^n \mathrm{d}z$$

Vertical structure of a self-gravitating polytropic disk

$$d\left(\frac{\mathrm{d}\psi_0}{\mathrm{d}z}\right) = 4\pi G\rho_{00}\Theta^n \mathrm{d}z \qquad d\mu = \Theta^n \frac{\mathrm{d}z}{H}$$

$$\mu = \frac{1}{4\pi G\rho_{00}H}\frac{\mathrm{d}\psi_0}{\mathrm{d}z} = -\frac{1+n}{2}H\frac{\mathrm{d}\Theta}{\mathrm{d}z} = -\frac{1+n}{2}\Theta^n\frac{\mathrm{d}\Theta}{\mathrm{d}\mu}$$

- Integrating from μ=0 (Θ = 0), we obtain

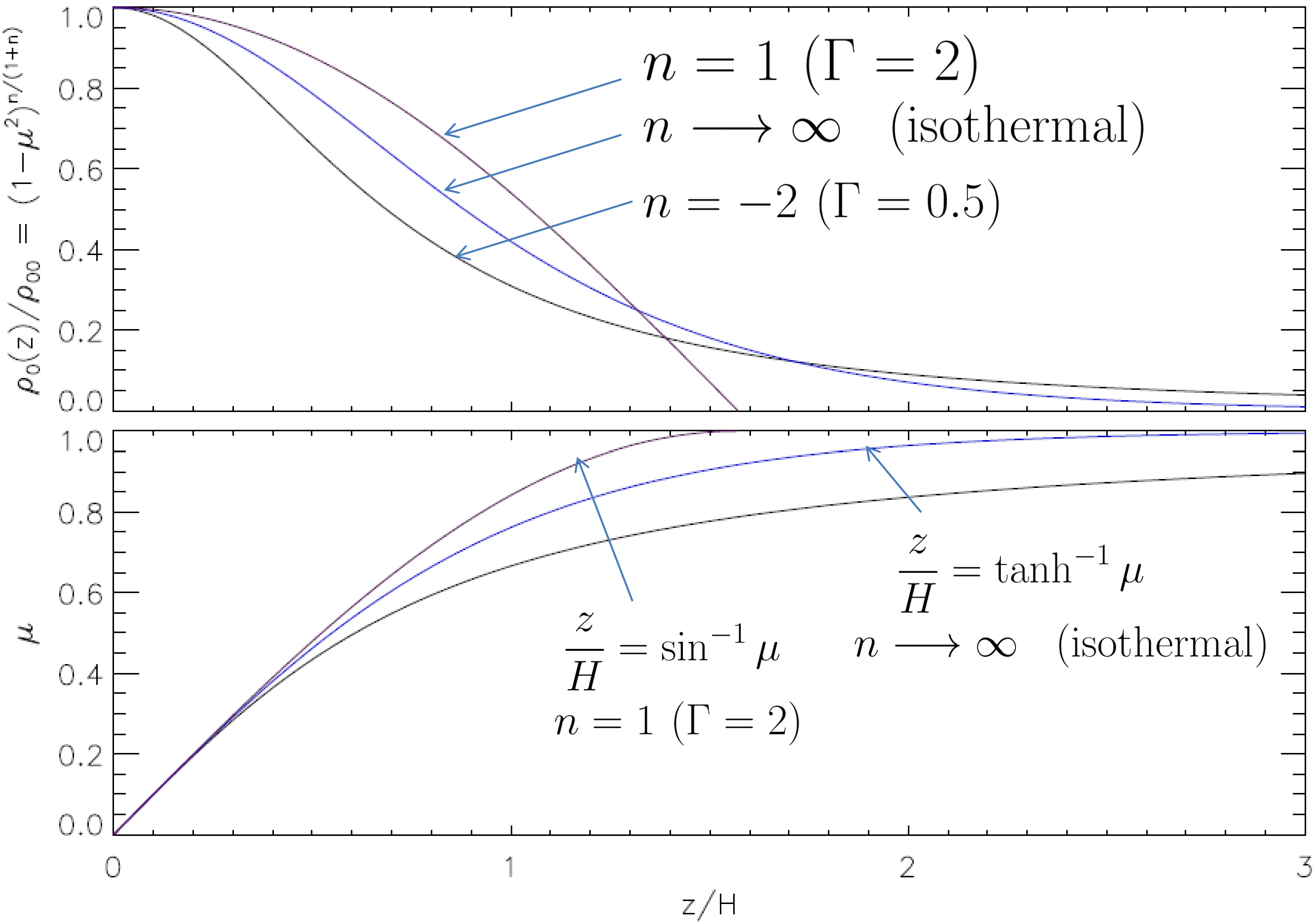
$$\mu = \frac{1}{4\pi G\rho_{00}H}\frac{\mathrm{d}\psi_0}{\mathrm{d}z} = -\frac{1+n}{2}H\frac{\mathrm{d}\Theta}{\mathrm{d}z} = -\frac{1+n}{2}\Theta^n\frac{\mathrm{d}\Theta}{\mathrm{d}\mu}$$

$$\Theta^n = (1-\mu^2)^{n/(1+n)}$$

$$\frac{z}{H} = \mu\, {}_2F_1\left(\frac{1}{2}, \frac{n}{n+1}; \frac{3}{2}; \mu^2\right)$$

Enable us to do analytic study

(Kim, 1990, undergraduate thesis)


$\rho_0(z)/\rho_{00} = (1-\mu^2)^{n/(1+n)}$
$n = 1\ (\Gamma = 2)$
$n \longrightarrow \infty$ (isothermal)
$n = -2\ (\Gamma = 0.5)$
μ
$\frac{z}{H} = \sin^{-1}\mu$
$n = 1\ (\Gamma = 2)$
$\frac{z}{H} = \tanh^{-1}\mu$
$n \longrightarrow \infty$ (isothermal)
z/H

Density structure

- Density structure
  - Polytropic index $\Gamma \equiv 1 + 1/n$ determines the stiffness of layer
  - Disks with $\Gamma > 1$ have finite extent
  - Disks with $\Gamma \leq 1$ extend to infinity ($z \rightarrow \infty$ as $\mu \rightarrow 1$)

- Stiffness polytropic EOS
  - Stiff EOS: large Γ
  - **Small changes in density provide strong pressure support against gravity**
  - In the limit of large Γ, the structure resembles that of a homogeneous incompressible disk

Pressure confinement by rarefied external medium

- Assume that the layer is truncated and bounded by an external presssure P_{ext} at $z = \pm a$ ($\mu = \pm A$)

$$P_{\rm ext} = K\rho_{00}^{1+1/n}(1 - A^2) = P_0(\mu = A)$$

- The total vertical column density

$$\Sigma(< a) = \int_{-a}^{+a} \rho_0 \mathrm{d}z = 2\rho_{00} H A$$

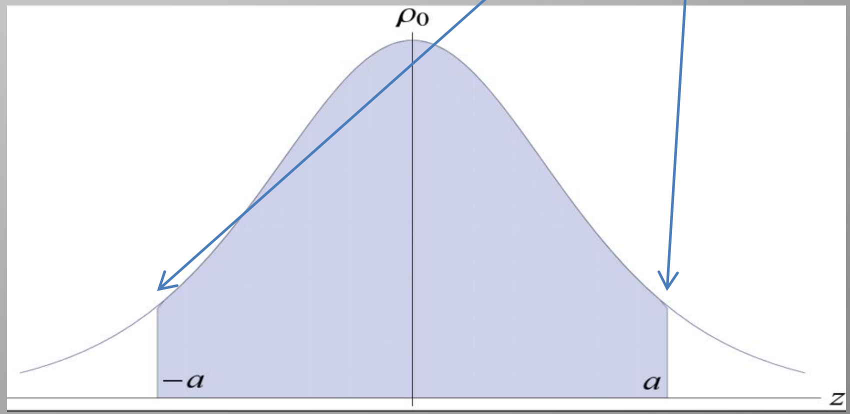
(AH : effective thickness of layer)

- Parameter A represents the degree of pressure confinement

$$A^2 = \frac{1}{1 + \frac{2P_{\rm ext}}{\pi G \Sigma_0^2}}$$

- A strongly compressed layer ($P_{ext} >> G\Sigma_0^2$, $A \rightarrow 0$) has nearly uniform density structure

Cut-offs at $z = \pm a$



Method of Linear Analysis

Linearization

- Linearized equations are

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \qquad \Longrightarrow \qquad \frac{\partial \hat{\rho}_1}{\partial t} + \nabla \cdot (\rho_0 \hat{\mathbf{u}}_1) = 0$$

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho}\nabla P - \nabla \psi + 2q\Omega_0^2 x \hat{\mathrm{x}} - 2\mathbf{\Omega_0} \times \mathbf{u}$$

$$\Longrightarrow \frac{\partial \hat{\mathbf{u}}_1}{\partial t} = \frac{\hat{\rho}_1}{\rho_0^2}\nabla P_0 - \frac{1}{\rho_0}\nabla \hat{P}_1 - \nabla \hat{\psi}_1 + q\Omega_0 \hat{u}_{1x}\hat{\mathbf{y}} - 2\mathbf{\Omega_0} \times \hat{\mathbf{u}}_1$$

$$\nabla^2 \psi = 4\pi G \rho \qquad \Longrightarrow \qquad \nabla^2 \hat{\psi}_1 = 4\pi G \hat{\rho}_1$$

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{P}{\rho^\gamma}\right) = 0 \qquad \Longrightarrow \qquad \hat{P}_1 = c_s^2 \hat{\rho}_1 \qquad c_s^2 = \gamma \frac{P_0}{\rho_0} \text{ with } \gamma = \Gamma$$

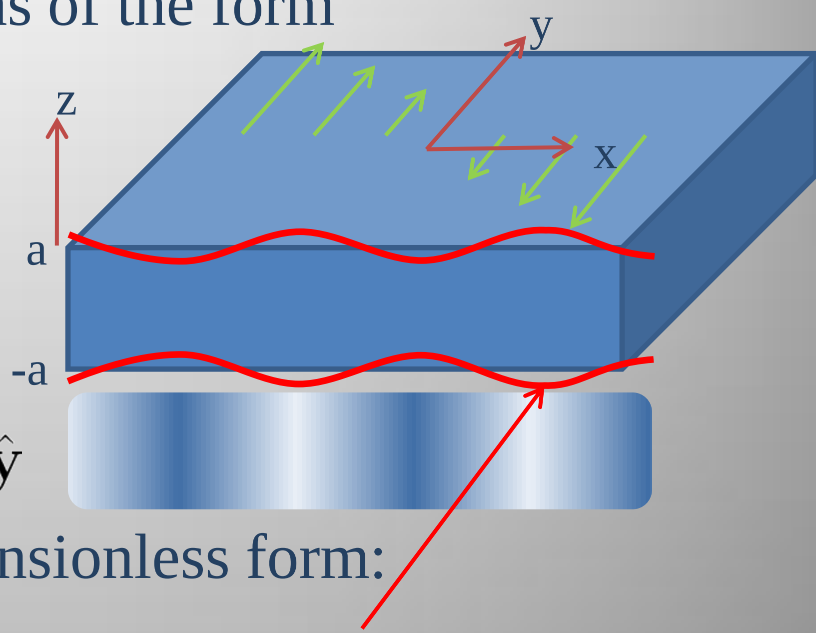
- Note that we do not consider convective mode
  - Polytropic index Γ = 1 + 1/n = γ adiabatic index

Axisymmetric perturbation

- Consider axisymmetric perturbations of the form

$$\hat{w}_1 = w_1(z)\exp(i\omega t + ikx)$$

$$\frac{\partial}{\partial t} \to i\omega \qquad \frac{\partial}{\partial x} \to ik$$



- It is convenient to introduce h_1, ξ_1

$$h_1 = P_1/\rho_0 \qquad \mathbf{u}_1 = i\omega\xi_1 + \xi_{1x} q\Omega_0 \hat{\mathbf{y}}$$

and express the quantities in a dimensionless form:

$$\psi_1/c_s^2(0),\ h_1/c_s^2(0) \to \psi_1,\ h_1$$

$$\rho_1/\rho_{00} \to \rho_1 \qquad z/H \to z$$

$$\frac{H^2}{c_s^2(0)}\kappa_0^2,\ \frac{H^2}{c_s^2(0)}\omega^2 \to \kappa_0^2,\ \omega^2$$

Note that we only consider symmetric deformation of surfaces

$$\xi_{1z}(x, z = a, t = 0) = \xi_{1z}\big|_a \cos(kx)$$

Perturbation equations in non-dimensional form

- After non-dimensionalization, we obtain
  - A set of equations that is equivalent to a set of four 1st order ODE

$$\Theta^n \frac{\mathrm{d}}{\mathrm{d}\mu}\xi_{1z} = \frac{2\mu}{\gamma\Theta}\xi_{1z} + \left(\frac{k^2}{\omega^2-\kappa_0^2} - \frac{1}{\Theta}\right)h_1 + \frac{k^2}{\omega^2-\kappa_0^2}\psi_1$$

$$\Theta^n \frac{\mathrm{d}}{\mathrm{d}\mu}h_1 = \omega^2\xi_{1z} - \Theta^n\frac{\mathrm{d}\psi_1}{\mathrm{d}\mu}$$

$$\Theta^n \frac{\mathrm{d}}{\mathrm{d}\mu}\Theta^n\frac{\mathrm{d}}{\mathrm{d}\mu}\psi_1 = \frac{2}{\gamma}\Theta^{n-1}h_1 + k^2\psi_1$$

- Given A, k, and κ_0^2 , we numerically solve for ω^2 along with appropriate boundary conditions

Boundary and symmetry conditions

- Free-surface boundary condition at z = a (μ = A)
  - Lagrangian pressure perturbation vanishes

$$\Delta P(z=a) = P_1(a) + \xi_{1z}\big|_a \frac{\mathrm{d}P_0}{\mathrm{d}z}(a) = 0 \quad\Longrightarrow\quad h_1(a) = \frac{2A}{\gamma}\xi_{1z}|_a$$

- Potential at z = a
  - Gauss's theorem gives potential difference

$$\frac{\mathrm{d}\psi_1}{\mathrm{d}z}\bigg|_a + k\psi_1(a) = -4\pi G\rho_0(a)\xi_{1z}$$

$$\Longrightarrow\quad \frac{\mathrm{d}\psi_1}{\mathrm{d}z} + k\psi_1 = -\frac{2}{\gamma}(1-A^2)^{n/(n+1)}\xi_{1z}|_a$$

- We only consider symmetric modes(sausage-type) which are subject to GI

$$\frac{\mathrm{d}\psi_1}{\mathrm{d}z} = \frac{\mathrm{d}h_1}{\mathrm{d}z} = 0 \quad \text{at} \quad z = 0$$

Numerical Method

- Aim to compute dispersion relation $\omega^2 = \omega^2(k; A; \kappa_0^2)$
- Given k, A, κ_0^2
- Fix $\xi_{1z}|_a=1$ and guess $\psi_1|_a$, ω^2 to obtain the values of h_1, ξ_{1z}, ψ_1, and $d\psi_1/dz$ at $\mu = A$ using the boundary conditions
- Integrate the equations using RK4 method from $\mu = A$ to $\mu = 0$
- Check whether the symmetry condition holds, i.e.,

$$\frac{\mathrm{d}\psi_1}{\mathrm{d}z} = \frac{\mathrm{d}h_1}{\mathrm{d}z} = 0 \quad \text{at} \quad z = 0$$

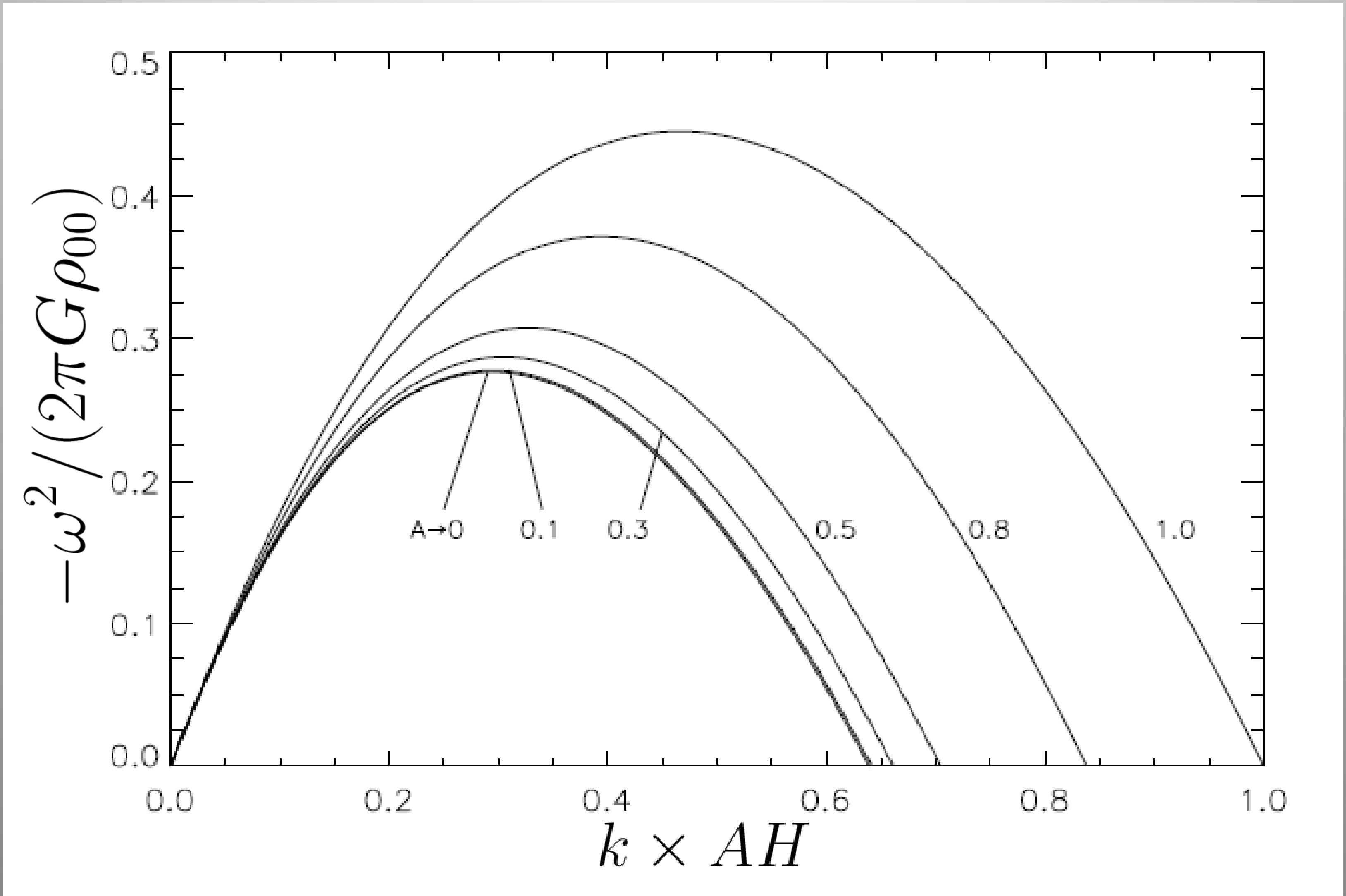
- Equivalent to finding a zero of a mapping

$$F : (\psi_1|_a, \omega^2) \to (dh_1/dz, d\psi_1/dz)$$

- Numerically compute derivatives(Jacobian) using different guesses
  - (Newton-Rhapson's method)
- Solutions corresponding to nearby points obtained in succession

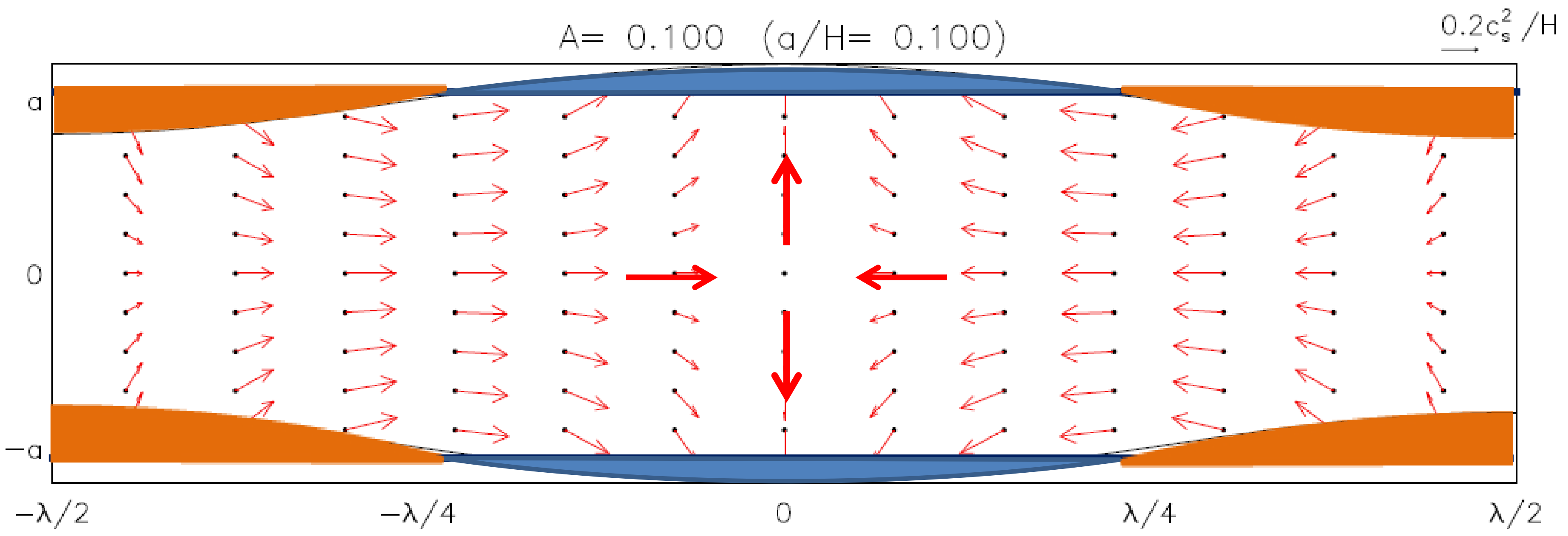
Results of Linear Analysis (Isothermal only)

Dispersion relation (non-rotating isothermal case)



Unstable mode of pressure-confined disk

- It is gravitationally unstable: gravity defeats pressure gradient force
- While lateral streamlines produce negative horizontal velocity gradient at the center (convergent), the behavior of fluid is nearly incompressible as it is counterbalanced by positive vertical velocity gradient (divergent)
- Pertubed gravity is due almost entirely to surface displacement of gravitating matter at $z = \pm a$



Distortional type of GI

- The full perturbed surface density consists of two parts

$$\Sigma_1 = \int_{-a-\xi_{1z}|_a}^{-a} \rho_0 \mathrm{d}z + \int_{+a}^{+a+\xi_{1z}|_a} \rho_0 \mathrm{d}z + \int_{-a}^{+a} \rho_1 \mathrm{d}z$$

Due to surface distortion Density perturbation

- It can be analytically shown that the gravitational potential due to the first two terms on the RHS is (for symmetric mode)

$$\psi_{\mathrm{surf}} = -\frac{4\pi G \rho_0(a) \xi_{1z}|_a}{k} e^{-ka} \cos(kx) \cosh(kz)$$

- Note that incompressible fluid has only the first two terms
- For disks with 0 < A < 1, we can distinguish two different source of the perturbed gravity using the above equation from the numerically obtained eigenfunction ψ_1

$$\psi_1 = \psi_{\mathrm{surf}} + \psi_{\mathrm{comp}}$$

Gravitationally unstable surface gravity wave

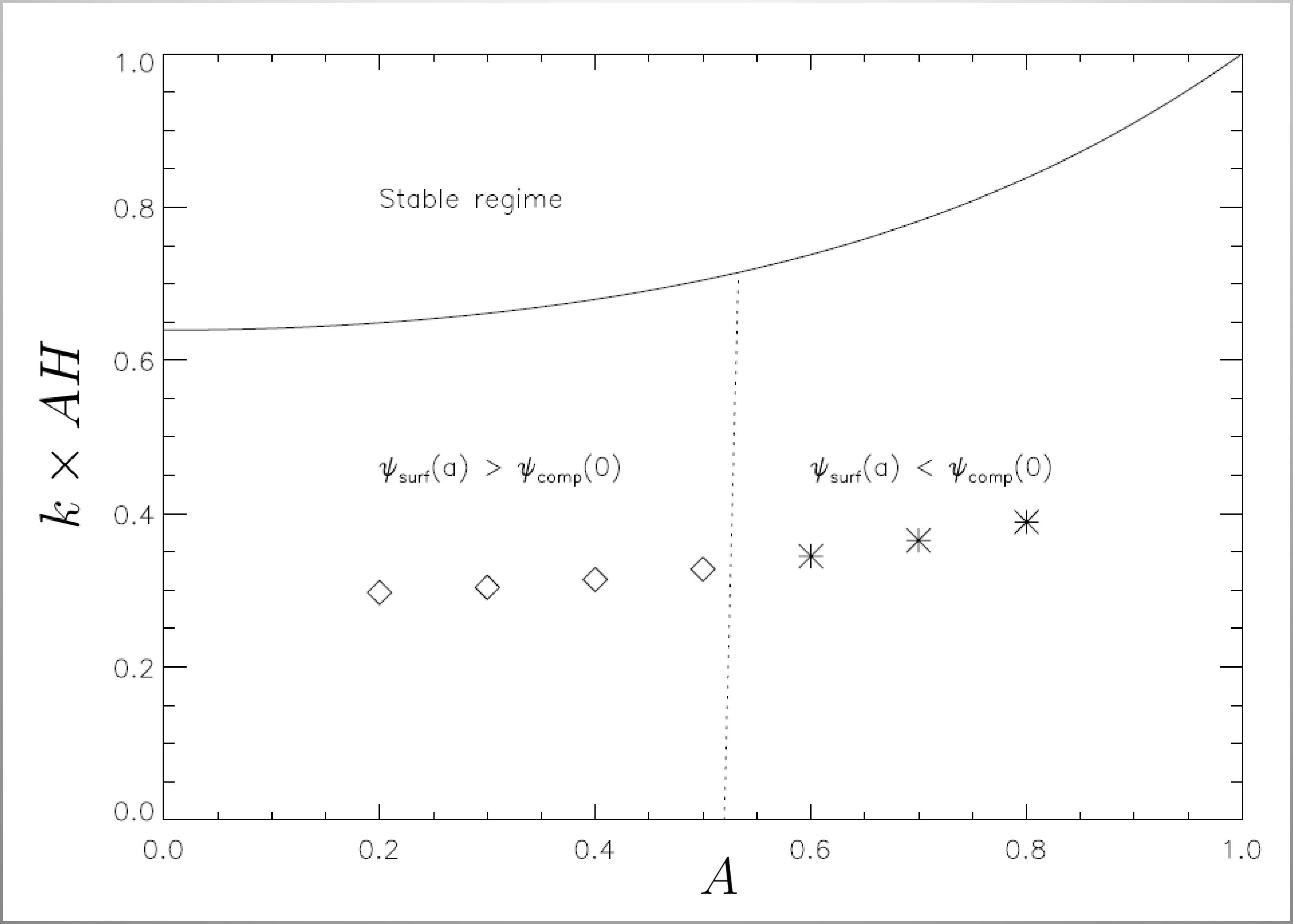
- In 3D disks, p, r, g, and f-mode can occur
  - p-mode: density perturbations due to compressibility, sound wave
  - r-mode: Coriolis force, inertial wave
  - g-mode: buoyancy causes convective motion (not explored in this study)
  - f-mode: surface gravity wave like surface gravity waves in the ocean
- Jeans instability is due to compressibility
- For a non-rotating incompressible gas disk whose both surfaces are allowed to move freely, only f-mode is excited
  - Only symmetric f-mode is unstable
- Dispersion relation for A ⟶ 0 (or incompressible disk) is

$$\frac{\omega^2}{4\pi G\rho_{00}} = \boxed{ka\tanh(ka)} \boxed{-\frac{1-e^{-2ka}}{2}}$$

gravity due to surface distortion

- (Compare the above case with ocean surface waves) $\omega^2 = gk\tanh(ka)$

Nature of instability(isothermal non-rotating case)



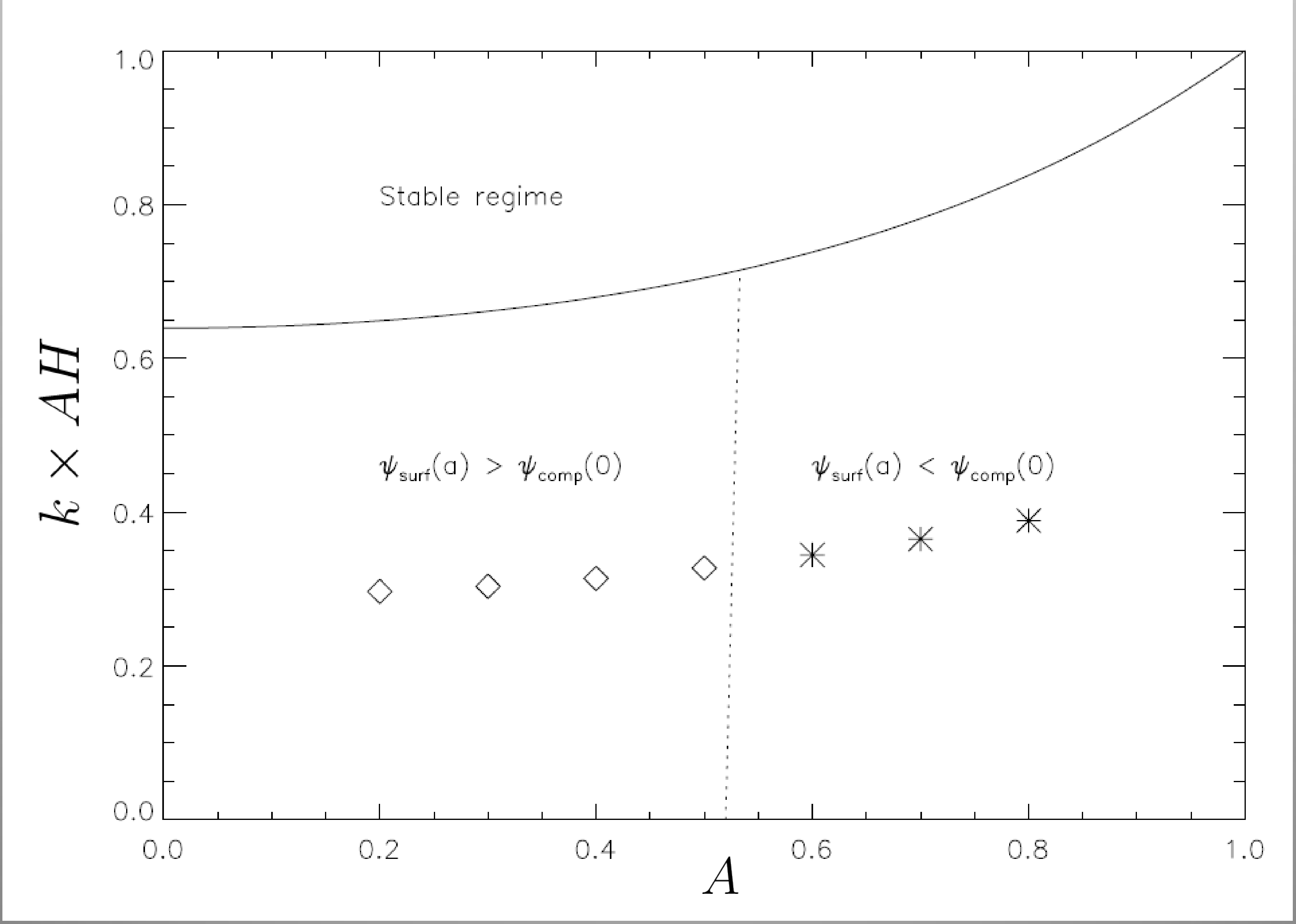
Comparison with simulations by Umekawa et al. (1999)

- 2D hydrodynamic simulations of a self-gravitating isothermal slab with pressure confinement
- Linearly growing stage: agrees well with the predictions from linear theory
- Nonlinear stage
  - A ≤ 0.5 fragment into stable clumps, the mass of which smaller than the Jeans mass
  - A ≥ 0.6 the Jeans instability creates dense collapsing clumps
- In concordance with the criterion given by linear analysis

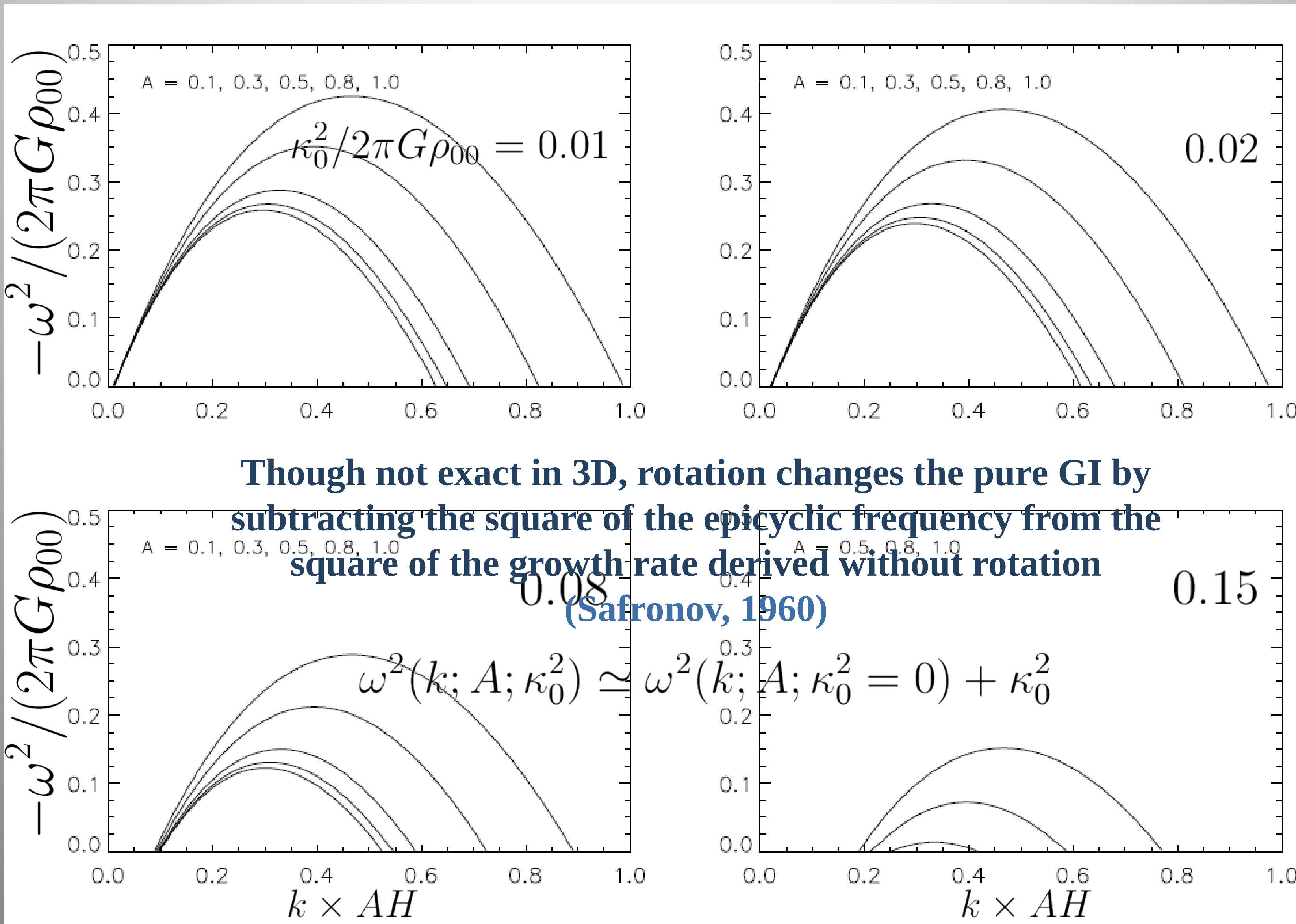
Table 2. The results of simulations.*

Model name	A	W/λ_{max}	ρ_{max}/ρ_{00}	v_{max}/c_s	t_f
A_2	0.2	1.0	1.28	0.47	12.71 (stable)
A_3	0.3	1.0	1.90	0.89	14.60 (stable)
A_4	0.4	1.0	3.97	1.47	17.50 (stable)
A_5	0.5	1.0	11.61	2.21	23.07 (stable)
A_6	0.6	1.0	161.14	3.31	16.35 (collapse)
A_7	0.7	1.0	775.40	4.44	13.34 (collapse)
A_8	0.8	1.0	789.22	6.80	12.43 (collapse)
W_3	0.3	4.0	249.16	11.14	29.85 (collapse)

Nature of instability(isothermal non-rotating case)



Effect of rotation



Though not exact in 3D, rotation changes the pure GI by subtracting the square of the epicyclic frequency from the square of the growth rate derived without rotation

(Safronov, 1960)

$$\omega^2(k; A; \kappa_0^2) \simeq \omega^2(k; A; \kappa_0^2 = 0) + \kappa_0^2$$

Stability criterion - Toomre's Q parameter

- Local dispersion relation for axisymmetric perturbation in a rotating razor-thin disk

$$\omega^2 = c_s^2 k^2 - 2\pi G \Sigma_0 |k| + \kappa_0^2$$

- The stability properties of gas disks are often expressed in terms of the Toomre Q-parameter (Toomre 1964)

$$Q = \frac{c_s \kappa_0}{\pi G \Sigma_0}$$

$Q < 1$: the disk is vigorously unstable and can fragment into self-gravitating clumps

- $Q_c = 0.676$ for isothermal gas disk with vertical stratification (Goldreich & Lynden-Bell, 1965, Kim & Ostriker, 2002)

Stability criterion for a pressure-confined self-gravitating layer

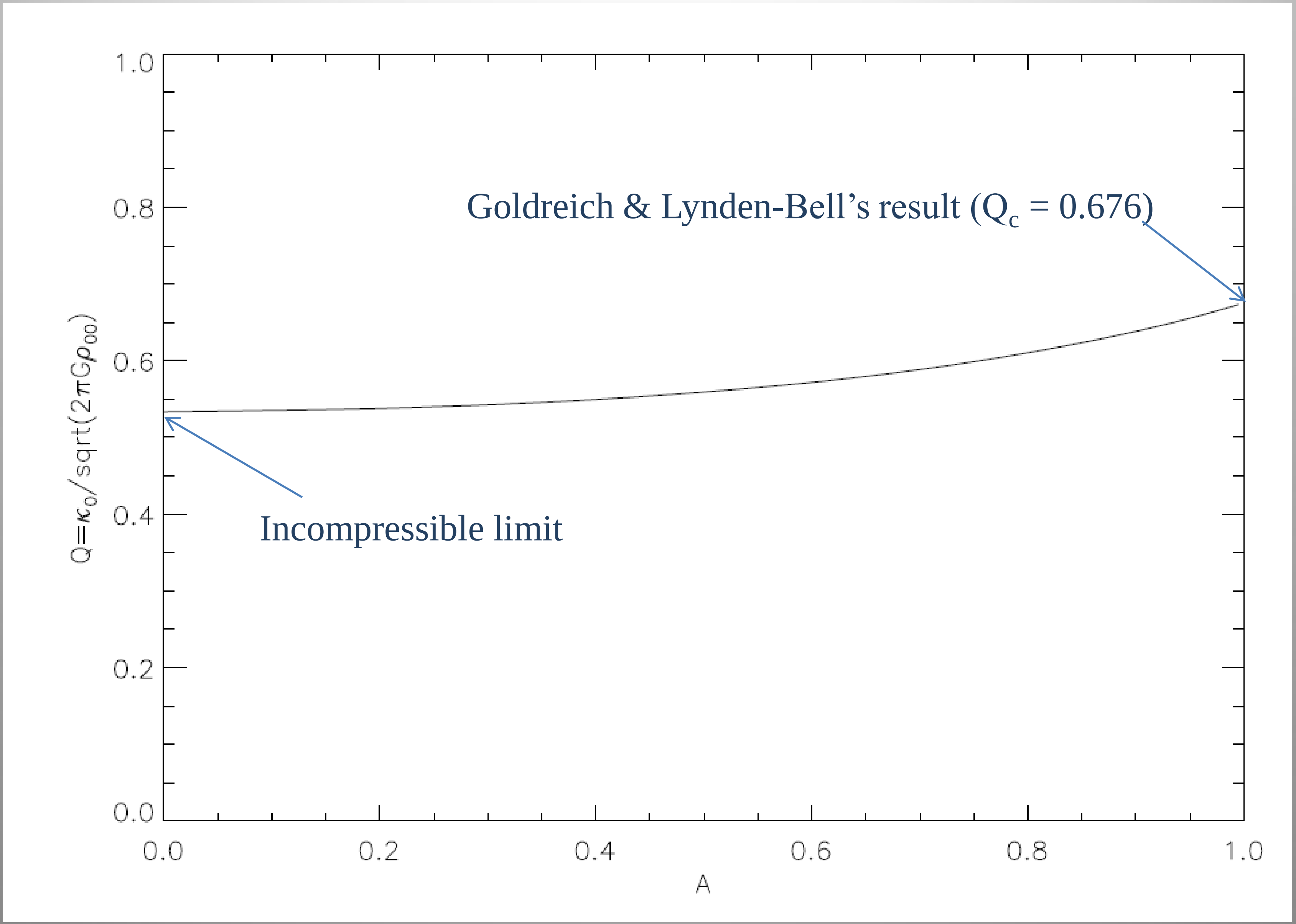
- Recall that $\Sigma_0 = 2\rho_{00}HA$, if we use Toomre's Q:

$$Q = \frac{c_s\kappa_0}{\pi G\Sigma_0} = \frac{\kappa_0}{2\pi G\rho_{00}}\frac{c_s}{AH} = \frac{1}{A}\cdot\frac{\kappa_0}{\sqrt{2\pi G\rho_{00}}}$$

- We require $\kappa_0^2 \sim G\rho00$ for stability regardless of the strength of external pressure
- Q diverges for distortional type of instability
- It is more natural to define Q` as a stability criterion

$$Q' = \frac{\kappa_0}{\sqrt{2\pi G\rho_{00}}}$$

- For pressure-confined disks, we need compare only two parameters because the sound speed of the disk is determined by hydrostatic condition



Goldreich & Lynden-Bell's result ($Q_c = 0.676$)
Incompressible limit
$Q=\kappa_0/\mathrm{sqrt}(2\pi G\rho_{00})$
A
1.0
0.8
0.6
0.4
0.2
0.0

Summary

- Pressure confinement by a rarefied external medium
  - Reduces the ratio of disk sound-crossing time (~ a/c_s) to gravitational free-fall time ($\sim 1/\sqrt{G\rho_{00}}$)
  - Thus determine the "hotnesss" of the disk
  - In the limit of strong compression, the disk has no vertical structure
- The nature of GI in compressed layer
  - Essentially due to surface distortion
  - Surface gravity wave can be unstable due to self-gravity
  - Jeans instability and distortional type of instability occur together for moderately confined layer
- Local axisymmetric stability of pressure
  - Rotation vertically shifts dispersion diagram

    $$\omega^2(k;A;\kappa_0^2) \simeq \omega^2(k;A;\kappa_0^2=0)+\kappa_0^2$$
  - Better to use stability parameter $Q` = \kappa_0/\sqrt{(2\pi G\rho_{00})}$