Pseudo Potential in Kerr Geometry and its Applications

Soumen Mondal\textsuperscript{1,2}

1. Korea Astronomy and Space Science Institute, Daejeon, Korea.
In astrophysical problems it is often advantageous to use pseudo-potentials instead of full general relativity. An well known example is so-called ‘Paczyn’ski-Wiita potential’ in which the gravitational potential of the star of mass $M$ namely, $\left[-\frac{GM}{r}\right]$ is replaced by

$$\Phi_{PW} = -\frac{GM}{r - \frac{2GM}{c^2}}$$

We propose an effective potential $\Phi_{eff}$ (gravitational pot. + centrifugal pot.) which adequately describes the space-time around a rotating black-hole. We call this as ‘Pseudo-Kerr potential’.

The form of our potential (in the units of $\left[\frac{GM_{BH}}{c^2}\right]$) is,

$$\Phi_{eff} = 1 - \frac{1}{r - r_0} + \frac{2a}{r^3} \left(\tilde{l} + \omega r^2\right) \frac{1}{\sin \theta} + \frac{\alpha^2}{2r^2} \left(\tilde{l} + \omega r^2\right)^2$$

where, $a$ is Kerr Parameter,

$$r_0 = 0.04 + 0.97a + 0.085a^2,$$

$$\alpha^2 = \frac{r^2 - 2r + a^2}{r^2 + a^2 + \frac{2a^2}{r}}$$

where, $\tilde{l} = -u_\phi$ is the conserved specific angular momentum for the particle whereas $l = -u_\Phi/u_t$ is the conserved specific angular momentum for fluid. Symbol $u_\mu$ being the covariant components of the four velocity.
1. **Viscous Effect:**
Fluid Dynamics in the presence of viscosity to study shock waves, 2009 MNRAS.
in Collaboration with Dr. Prasad Basu (Indian Institute of Science, India) and Prof. S.K. Chakrabarti.

2. **Smooth Particle Hydrodynamics Simulations:**
S.P.H simulations in 1-Dimension and in 2-Dimensions.
in Collaboration with Prof. Diego Molteni (Dipartimento di Fisica e Tecnologie Relative, Palermo, Italy) and Prof. S.K. Chakrabarti.

3. **Gravitational Wave:**
Emission of gravitational wave from a binary black hole in the presence and in the absence of accretion disk, 2008 MNRAS.
in Collaboration with Dr. Prasad Basu (Indian Institute of Science, India) and Prof. S.K. Chakrabarti.

4. **Radiation Hydrodynamics:**
Analysis of the spectrum for different Kerr parameter in the presence and in the absence of viscosity.
in Collaboration with Dr. Samir Mandal (Ben-Gurion University of the Negev Beer-Sheva, Israel) and Prof. S.K. Chakrabarti.
The Plan of the talk is the following

(1): Particle Dynamics in Pseudo-Kerr Potential,

(2): Fluid dynamics: Accretion around Black holes,

(3): Shocks in Accretion Flows: SPH results, QPOs

(4): Conclusions.

The results obtained from our potential match with the general relativistic results for the values of Kerr parameter $(a)$ ranging from $-1$ to $0.8$. 
Effective Potential. ... Reference: MNRAS-06.

The curves are drawn for (from top to bottom) \(\tilde{l} = 3.2, (ms)3.4872(pk)3.4641(gtr), 3.6, 3.8, (mb)4.0201(pk)4.0(gtr), 4.2\) respectively for the Kerr parameter value \(a = 0.0\).
The curves are drawn for (from top to bottom) $\tilde{l} = 2.7$, (ms)2.8978(pk)2.9029(gtr), 3.1, 3.3, (mb)3.4237(pk)3.4142(gtr), 3.5 respectively.
Fig. 2(a-b): (a) The locations of the marginally bound ($r_{mb}$) and marginally stable ($r_{ms}$) orbits and (b) the angular momenta $l_{mb}$ and $l_{ms}$ at these orbits. Results generally agree for $-1 < a < 0.8$. 

Reference: IJP-05.
Marginally Stable Values of Effective Potential or Maximum Energy Dissipation with Kerr parameters ($a$).

Results generally agree for $-1 < a < 0.8$. 
The particle trajectories using pseudo-Kerr potential. The parameters are $a = 0.8, \tilde{l} = -4.5$. 

Reference: MNRAS-06.
Gravitational Wave Emission from binary black holes

Figure shows infall time decreases with increase of Kerr parameter.

Height of the Accretion Disk

The height of the disk in the vertical equilibrium (NT73) is given by

\[ h(r) = \left( \frac{p}{\rho_0} \right)^{\frac{1}{2}} r^{\frac{3}{2}} \frac{\Gamma \left( \frac{(r^2+a^2)^2 - \Delta a^2}{(r^2+a^2)^2 + 2\Delta a^2} \right)}{\Gamma} \]

where, \( \Gamma = \left[ 1 - \frac{A^2}{\Delta r^4} (\Omega - \omega)^2 \right]^{-\frac{1}{2}} \) and \( A = r^4 + r^2a^2 + 2ra^2 \)

In our case the vertical height of the disk is given by

\[ h(r) = \left( \frac{p}{\rho_0} \right)^{\frac{1}{2}} r^{\frac{1}{2}} \left( \frac{\partial \Phi_{eff}}{\partial R} \right)^{-\frac{1}{2}} \bigg|_{R=r} \quad \text{and} \quad R^2 = r^2 + z^2 \]
Height of the Accretion Disk

The height of the disk in the vertical equilibrium (NT73) is given by

\[ h(r) = \left( \frac{p}{\rho_0} \right)^{\frac{1}{2}} r^{\frac{3}{2}} \frac{1}{\Gamma} \left[ \frac{(r^2+a^2)^2-\Delta a^2}{(r^2+a^2)^2+2\Delta a^2} \right]^{\frac{1}{2}} \]

where,

\[ \Gamma = \left[ 1 - \frac{A^2}{\Delta r^4} (\Omega - \omega)^2 \right]^{-\frac{1}{2}} \quad \text{and} \quad A = r^4 + r^2 a^2 + 2ra^2 \]

In our case the vertical height of the disk is given by

\[ h(r) = \left( \frac{p}{\rho_0} \right)^{\frac{1}{2}} r^{\frac{1}{2}} \left( \frac{\partial \Phi_{eff}}{\partial R} \right)^{-\frac{1}{2}} \bigg|_{R=r} \quad \text{and} \quad R^2 = r^2 + z^2 \]
(2): Fluid Dynamics

- **Basic Equations:** (Sonic point analysis, nature of the sonic points and different solutions)
- **Parameter Space:** (for different Kerr parameter.)
- **Topology of the Flow:** (accretion and wind topology.)
- **Shock Waves:** (a] the region of the parameter’s space responsible for the formation of shock waves and [b] the variation of the location of shock waves with Kerr parameter and viscosity of the flow.)
The Rankine-Hugoniot conditions across the shock are given by

1. \( \mathcal{E}_+ = \mathcal{E}_- \) (Energy)

2. \( \dot{M}_+ = \dot{M}_- \) (Mass flux)

3. \( P_+ + \rho_+v_+^2 = P_- + \rho_-v_-^2 \) (Total Pressure).
Importance of the shock in accretion flow

• The hot puffed up post-shock matter intercepts and inverse-Comptonizes a significant portion of the soft photons from the disk, to produce high energy power law tail.

• Excess thermal gradient force in the post-shock drives a part of the incoming matter as outflow/jet along axis.

• Origin of QPO can be explain by shocks.

• It can produce non-thermal electrons through shock acceleration, which is essential to explain the non-thermal power-law spectrum in high energy.

Reference:
Variation of Energy with the Sonic points

\[ \mathcal{E} = \left( 1 + \frac{1}{2n+1} \right) \frac{\Phi'_{\text{eff}}(r,l)}{\frac{2}{(2n+1)} \frac{H'}{H}} + \Phi_{\text{eff}}(r,l). \]

The curves are drawn for (from top to bottom) \( l = 2.3, 2.488, 2.7, 2.9, 3.1, 3.3, 3.5, 3.7 \) respectively.
Variation of Sonic points with Energy and Ang. mom.

Variation of energy with distances

Variation of sp. ang. mom. with distance
Variation of Sonic points with Energy and Ang. mom.
Flow Topologies at various regions of the Parameters Space

Region of the Parameters Space available Standing and Oscillating Shocks.
Region of the Parameters Space available Shocks in Accretion flow.
Parameters space contains shocks in viscous accretion flow.

The different regions of the parameters space shrink as the viscosity parameter $\alpha_{\Pi}(=0.0, 0.05, 0.1)$ increases and moved up to the high energy regions. The parameter spaces also shift towards the region of higher energy and lower angular momentum with the increase of Kerr parameter while their overall-shape remain almost unchanged.
Shocks in Accretion and Wind flows.

The solution topology (arrowed curve) includes a standing shock.
Shocks in Accretion and Wind flows.

The solution topology includes a standing shock.
Fig. 2(a): Variation of the global solution topologies with the viscosity parameter $\alpha_{\Pi}$ of the accretion flow are shown for a moderate value of kerr parameter ($a=0.5$). The other required parameters are also mentioned above.

Fig. 2(b): Variation of the topologies with the viscosity parameter $\alpha_{\Pi}$ for same values of kerr parameter but different values of initial angular momentum and inner sonic point (mentioned above).
General Questions

Q1: How does the shock location vary with

A. the flow parameter?

B. the spin of the black hole?

C. the viscosity of the flow?
The variation of shock location with energy of the flow ($\alpha_{II} = 0$).

(a) Accretion shock (Solid Curve),
(b) Winds shock (Dashed Curve).

Shocks form at a very low angular momentum for $\alpha=0.5$. 
Variation of Shock Locations with Kerr parameter

Shock Location vs Energy for different Kerr parameters:
- $a = 0.55$
- $a = 0.50$
- $a = 0.45$

Shocks in accretion and wind are indicated in the graph.

Angular momentum $= 2.95$
Shock locations shift rapidly towards BH for higher values of Kerr parameter ($a$).
Extreme shock locations Viscosity and Spin.

Extreme shock forms close to the horizon for rapidly rotating BH but move outwards with the increase of viscosity.
General Questions

Q2: Is there any correlation between QPO frequency and spin of the black hole?

If yes, then

Q3: Is it possible to estimate the spin of the black hole from its QPO frequency?
SPH simulation produces Shocks at different time steps

Smooth Particle Hydrodynamics Simulation shows Standing Shocks in the accretion flow
SPH simulations consistent with analytic solution.

Shocks at different time steps.
SPH simulations produce Oscillation of Post-shock matter
Summary:
Important results and conclusions:

• Accretions/winds occur at lower values of angular momentum and higher energy as spin of the black hole increases.

• Flow variables such as, shock locations, sonic points etc. shift rapidly towards the black hole for highly spinning Kerr black hole.

• Extreme shocks occur in case of inviscid flow. Shocks moves outwards with the increase of viscosity of the flow.

• These extreme shocks may important to introduce oscillations in the disk flow very close to the centre and therefore, may excite QPOs mechanism observed in many Galactic Micro-quasars.
References:

- Mukhopadhyay, B. 2002, APJ, 581, 427

THANK YOU.